Existence of energy minimizers in homotopy classes; the metric space story

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Background

In the mid 60's J. Eells and J. Sampson proved that given a C^1-map f from a compact Riemannian manifold to a compact nonpositively curved manifold, one can deform it continuously to a harmonic map. This harmonic map minimizes the Dirichlet 2-energy in the homotopy class of f.

- The assumption of nonpositively curved target is essential. In fact for classical homotopy the result fails without it.
- In 1993 N. Korevaar and R. Schoen generalized the result to hold when the target space Y is a nonpositively curved metric space (NPC space).
- In the mid to late 90's J. Jost studied a related problem of minimization among equivariant maps, allowing the source space to be a singular metric space.
- All of the above are concerned with harmonic case (i.e. Dirichlet p-energy, $p = 2$.)

Setting

We want to study the minimization question for maps in the *Newton Sobolev space* $N^{1,p}(X;Y)$ where

- X is a compact p-Poincare space with a doubling measure, Y is a compact NPC space, and $p \in (1, \infty)$;
- p-harmonic in this generality means minimizer of the p-energy $e_p(u) = \int_X g_u^p \mathrm{d}\mu$ in a given "homotopy class" of $N^{1,p}(X;Y)$.
- g_u is the *minimal p-weak upper gradient* of u, the substitute for $|Du|$ in the case of smooth spaces.

(p-weak) upper gradients

Let $u : (X, d, \mu) \to Y$ be a map from a metric measure space X to a metric space Y such that $u(x) \in Y$ for μ-a.e. x. A Borel function $g : X \to [0, \infty]$ is an *upper gradient* of u if

$$d_Y(u(\gamma(1)), u(\gamma(0))) \leq \int_\gamma g$$

for every rectifiable curve $\gamma : [0, 1] \to X$. If $u(\gamma(1))$ or $u(\gamma(0))$ are not in Y we interpret the LHS to be infinity.If the inequality holds for every rectifiable curve apart from an exceptional family Γ_0 (of curves) with

$$\mathrm{Mod}_p(\Gamma_0) = 0,$$

we say that g is a p-weak upper gradient of u.

Fact

If a map u has a p-integrable p-weak upper gradient then there is a pointwise minimal p-weak upper gradient, denoted g_u.

p-Poincare spaces

Poincaré inequality

A metric measure space (X, d, μ) supports a p-Poincaré inequality, $p \in [1, \infty)$, if there are constants $C, \sigma \geq 1$ so that

$$\fint_B |u - u_B| \mathrm{d}\mu \leq C \left(\fint_{\sigma B} g^p \mathrm{d}\mu \right)^{1/p}, \text{ for all } B,$$

$u \in L^1$, g upper gradient of u.

We assume in addition that the measure μ is doubling, i.e. there is a constant C such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)), \quad x \in X, r > 0.$$

The Newton Sobolev space

Recall the Kuratowski embedding $\iota : Y \to \ell^\infty(Y)$.

The Newtonian space $N^{1,p}(X; Y)$ consists of equivalence classes of p-integrable maps $u\colon X \to \ell^\infty(Y)$ for which $u(x) \in \iota(Y)$ for a.e. x and there exists a p-integrable p-weak upper gradient g. The equivalence relation is $u \sim v$ if $g_{u-v} = 0$ a.e.

It turns out that $u \sim v$ iff the set $\{u \neq v\}$ has zero *p-capacity* (we say that $u = v$ p-quasieverywhere). Here

$$\mathrm{Cap}_p(E) = \inf \left\{ \int_X g_f^p \mathrm{d}\mu : f \geq \chi_E \right\}, \ E \subset X$$

Capacity turns out to be very useful for describing some properties of Newtonian maps.

Quasicontinuity and p-quasicontinuity

Quasicontinuity is in the spirit of *almost continuity* for measurable maps.

Fact (Shanmugalingam '01, Björn-Björn '13)

Any Newtonian map $f \in N^{1,p}(X;Y)$ (from a compact Poincaré space with doubling measure) is p-quasicontinuous, meaning for every $\varepsilon > 0$ there is an open set $E,\ \mathrm{Cap}_p(E) < \varepsilon$ such that $f|_{X\setminus E}$ is continuous.

The notion of homotopy corresponding naturally to continuity-concept is the following.

Definition (p-quasihomotopy)

Two maps $u, v : X \to Y$ are p-quasihomotopic if there is a map $H : X \times [0,1] \to Y$ so that for each $\varepsilon > 0$ there exists an open set $E \subset X$ with $\mathrm{Cap}_p(E) < \varepsilon$ so that $H|_{X\setminus E\times[0,1]} : X \setminus E \times [0,1] \to Y$ is a continuous homotopy between $u|_{X\setminus E}$ and $v|_{X\setminus E}$.

Existence of minimizers

The hope is to obtain an existence result for energy minimizers in this generality.The proof of the following main result uses some geometric group theory and for it to work one needs to assume an extra assumption on the fundamental group of the target.However at this point an extra assumption is needed.

Theorem (S.)

Let X be a compact p-Poincaré space with doubling measure, Y a compact NPC space and $p \in (1, \infty)$. Assume that $\pi(Y)$ is a hyperbolic group. Then every p-quasihomotopy class $[v]$ of a map $v \in N^{1,p}(X; Y)$ contains a map $u \in [v]$ of minimal p-energy:

$$e_p(u) = \inf_{w \in [v]} e_p(w).$$

p-quasihomotopy and lifts

The starting point of the proof is the known fact that two (continuous) maps into an NPC manifold are homotopic iff their induced homomorphisms are conjugate.

Fact (Schoen, Yau, Burstall, White..)

For $p \geq 2$ $W^{1,p}$-Sobolev maps between manifolds induce homomorphisms between the fundamental groups.

In our generality this fails so we formulate the starting point differently.

Lemma (Reduction to lifts, S.)

Let Y be an NPC space. Then there is a covering space $\phi : \widehat{Y}_{\mathrm{diag}} \to Y \times Y$ so that two maps $u, v \in N^{1,p}(X; Y)$ are p-quasihomotopic if and only if the map $(u, v) \in N^{1,p}(X; Y \times Y)$ admits a lift $h \in N^{1,p}(X; \widehat{Y}_{\mathrm{diag}})$. $\widehat{Y}_{\mathrm{diag}}$ is given by

$$\widehat{Y}_{\mathrm{diag}} = \widetilde{Y} \times \widetilde{Y} / \operatorname{diag} \pi(Y).$$

A compactness result

The next step is to obtain a compactness result.

Theorem (stability of lifts, S.)

Suppose Y is a separable geodesic space, $\phi : \hat{Y} \to Y$ is an isometric covering map such that $\phi_\sharp \pi(\hat{Y}) \leq \pi(Y)$ satisfies the subconjugacy condition and $(u_j) \subset N^{1,p}(X; Y)$ is a sequence of maps with uniformly bounded L^p-norms of upper gradients, each admitting a lift in $N^{1,p}(X; \hat{Y})$. If u_j converges in L^p to $u \in N^{1,p}(X; Y)$ then u admits a lift in $N^{1,p}(X; \hat{Y})$.

Corollary (Compactness of p-quasihomotopy classes, S.)

Suppose Y is a compact NPC space *with $\pi(Y)$ a hyperbolic group*, $u_j \in N^{1,p}(X; Y)$ a bounded sequence of maps in the p-quasihomotopy class $[v] \subset N^{1,p}(X; Y)$. Then there is a subsequence converging in L^p to a map $u \in [v]$.

Where the hyperbolicity comes in

Subconjugacy condition

A subgroup H of a group G satisfies the subconjugacy condition if, for every sequence $(g_n) \subset G$ there is an element $g \in G$ so that

$$\liminf_{n\to\infty} H^{g^n} := \bigcup_{k\geq 1} \bigcap_{n\geq k} H^{g_n} \leq H^g, \ H^g = g^{-1}Hg.$$

Lemma (S.)

Let G be a (finitely generated) torsion free hyperbolic group. Then $\operatorname{diag} G \leq G \times G$ satisfies the subconjugacy condition.

Fact

The fundamental group of a compact NPC space is a finitely generated torsion free group. It is hyperbolic if and only if the universal cover of Y does not contain a plane.

Remarks

- The proof of stability for lifts uses a notion of "$u_\sharp \pi(X, x_0)$" $\leq \pi(Y, u(x_0))$ for a Sobolev-Newton maps u (but not a homomorphism $u_\sharp : \pi(X) \to \pi(Y)$!)
- the subconjugacy condition is needed because $u_\sharp \pi(X, x_0)$ is not known to be finitely generated.
- The assumption of hyperbolicity ultimately stems from this fact.

Thank you!