Hajłasz spaces and their reflexivity properties

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1 Introduction

Analysis on metric spaces is a quite recent area in mathematics, and has been met with increasing interest since the last decades of the twentieth century. By then it had become apparent that many applications, especially in the theory of quasiconformal mappings, required results from classical analysis in new, non-smooth settings. However, the class of general metric spaces is clearly too large for the purposes of a general theory of, say, differentiability of Lipschitz functions. Thus it was that at an early stage attempts at generalizing some classical tools in analysis focused on subdomains or submanifolds of $\mathbb{R}^n$. For instance in [5] the authors got as far as defining *Besov spaces* for certain domains of $\mathbb{R}^n$ (these spaces are beyond the reach of the general theory presented in this thesis).

It soon became apparent that in order to avoid repeating similar arguments, a general theory was needed. Moreover, an intrinsic theory would reveal the properties truly essential for the development of a theory of differentiability. In a geometric language, the existence of an abundance of rectifiable curves between any two points in the given metric spaces turned out to play an essential role in such a theory. The concept of a Loewner space (see [17], Chapter 8) directly quantifies this idea while also generalizing some classes of domains in $\mathbb{R}^n$ where classical results of Sobolev spaces hold. Other essential properties, discovered earlier, were the doubling property of the space or of the measure in question. This was known to be related, roughly speaking, to the "finite dimensionality" of the space. In particular, Assouad's embedding theorem ([17], Theorem 12.1, p. 98) and a result of Konyagin and Vol´berg (later extended in [30]) provide a link between complete doubling spaces and Euclidean spaces.

Groundbreaking articles on the subject include [16], where the notion of an upper gradient is introduced and the notion of a Poincaré inequality is related to Loewner spaces, and [12], [31] and [2] where different Sobolev type spaces over the given metric space are defined. Shanmugalingam, in [31] in 2000, introduced what will in the sequel be referred to as Newtonian spaces, utilizing the notion of an upper gradient. Four years earlier Hajłasz had defined The Hajłasz-Sobolev space for arbitrary metric measure spaces, and proven very general embedding theorems in this context. For spaces satisfying the aforementioned essential properties of being complete and doubling, and supporting some Poincaré inequality (which has largely replaced the notion of the Loewner space) Shanmugalingam proved the coincidence of the Newtonian and Hajłasz-Sobolev spaces. An even more important advancement in this function-space approach to analysis on metric spaces was obtained by Cheeger in [2] in 1999, when he proved the reflexivity of a (yet another) Sobolev type space over a complete doubling metric space supporting a Poincaré inequality. This approach included the construction of a sort of differentiable structure for these spaces, hence transfering much of the first order calulus of $\mathbb{R}^n$ to the setting of these metric measure spaces. Shanmugalingam [31] proved that in the abovementioned context the Sobolev space of Cheeger coincides with the other two.

In subsequent years there has been much interest in the properties of metric spaces satisfying a Poincaré inequality as well as in the development of variational calculus in metric spaces. Also attempts at weakening the hypotheses under which the metric space admits a differential structure have been made, along with different approaches toward a definition of metric differentiability.

This exposition serves as an introduction to the approach described above, with an emphasis on the Sobolev space perspective.

2 Prerequisites

2.1 Some functional analysis

Functional analysis lies at the very heart of many of the proofs presented in this thesis and it is therefore advisable to have some backround knowledge of it. Some concepts that will be needed, however, can be considered sufficiently advanced (or rather specialized) so as to be beyond reasonable undergraduate level background. These concepts will be presented here, mostly without proof. Instead, references to material containing proofs are presented in connection to every theorem or lemma.

The following two theorems are elementary in nature and can be found in most textbooks in functional analysis, one such being ([8], p. 160). They will prove useful in section 4.

Lemma 2.1.1. *Let X be a reflexive Banach-space and Y a closed subspace of X. Then Y – as a Banach space with norm inherited from X – is also reflexive*

Lemma 2.1.2. *If X and Y are two Banach spaces which are isomorphic and X is reflexive then Y is also reflexive.*

The third result is the so called *Mazur's lemma* which is a consequence of a geometric version of the Hahn-Banach theorem.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a Banach space and $(x_n) \subset X$ a sequence that converges weakly to $x \in X$. Then there is a sequence of convex combinations of the members of the sequence, that is, a sequence*

$$y_k := \sum_{i=1}^{m_k} a_i x_{k_i}, \quad a_i \geq 0,\ a_1 + \cdots + a_{m_k} = 1$$

that converges to x in norm.

This convenient result enables one to construct strongly convergent sequences from weakly convergent ones. A proof can be found in ([32], p. 28).

Theorem 2.2. *Let $(X_k, \|\cdot\|_k)_{k\in K}$ ($K \subset \mathbb{N}$) be a sequence of Banach spaces and let $X := \bigoplus_{\ell^p(K)} X_k$, $1 < p < \infty$, be the Banach space of all sequences*

$$x = (x_k)_{k\in K}, \quad x_k \in X_k$$

for which the norm

$$\|x\|_X^p := \sum_{k\in K} \|x_k\|_k^p$$

is finite. Then the dual X^* of X is $\bigoplus_{\ell^{p'}(K)} X_k^*$. (Here p' is the Hölder conjugate exponent of p.) In particular if each X_k is reflexive, then so is X.

Proof. Let $\varphi : X \to \mathbb{R}$ be an element of X^*. Define $\varphi_k : X_k \to \mathbb{R}$ by $\varphi_k(x) = \varphi(0, \ldots, x, 0, \ldots)$ where x is in the kth slot. This gives, for every k, an element of X_k^* for which $\|\varphi_k\|_{X_k^*} \leq \|\varphi\|_{X^*}$. Now for every $x = (x_k)_{k\in K} \in X$ $\varphi(x)$ can be written as

$$\varphi(x) = \sum_{k\in K} \varphi_k(x_k),$$

since $(x_1, \ldots, x_k, 0, \ldots) \to x$ in X. Thus

$$|\varphi(x)| \leq \sum_{k\in K} |\varphi_k(x_k)| \leq \sum_{k\in K} \|\varphi\|_{X_k^*}\|x_k\|_k \leq$$
$$\left(\sum_{k\in K} \|\varphi_k\|_{X_k^*}^{p'}\right)^{1/p'} \left(\sum_{k\in K} \|x_k\|_k^p\right)^{1/p} = \left(\sum_{k\in K} \|\varphi_k\|_{X_k^*}^{p'}\right)^{1/p'} \|x\|_X,$$

yielding

$$\|\varphi\|_{X^*} = \sup\{|\varphi(x)| : \|x\|_X \leq 1\} \leq \left(\sum_{k\in K} \|\varphi_k\|_{X_k^*}^{p'}\right)^{1/p'}.$$

For the opposite inequality let $\varepsilon > 0$ be arbitrary and take for each k an element $x_k \in X_k$, $\|x_k\|_k \leq 1$ so that

$$\varphi_k(x_k)^{p'} + \varepsilon/2^k > \|\varphi_k\|_{X_k^*}^{p'}.$$

For this calculate

$$\sum_{k\in K} \|\varphi_k\|_{X_k^*}^{p'} < \sum_{k\in K} \varphi_k(x_k)^{p'} + \varepsilon = \sum_{k\in K} \varphi_k(x_k)^{p'-1}\varphi_k(x_k) + \varepsilon$$
$$= \varphi\left(\sum_{k\in K} \varphi_k(x_k)^{p'-1}x_k\right) + \varepsilon \leq \|\varphi\|_{X^*}\|y\|_X + \varepsilon \tag{2.1.1}$$

where $y = \sum_{k\in K} \varphi_k(x_k)^{p'-1}x_k$. Its norm can be estimated by

$$\|y\|_X^p = \sum_{k\in K} \varphi_k(x_k)^{(p'-1)p}\|x_k\|_k^p \leq \sum_{k\in K} \varphi_k(x_k)^{p'} \leq \sum_{k\in K} \|\varphi_k\|_{X_k^*}^{p'}.$$

Introducing the shorthand-notation

$$\sum_{k\in K} \|\varphi_k\|_{X_k^*}^{p'} = \|\varphi_k\|_{\ell^{p'}}^{p'}$$

equation (2.1.1) simplifies to

$$\|\varphi_k\|_{\ell^{p'}}^{p'} \leq \|\varphi\|_{X^*}\|\varphi_k\|_{\ell^{p'}}^{p'/p} + \varepsilon$$

for arbitrary ε. This readily implies the desired inequality. □

Next is a topic of high interest in itself. It is a condition the norm of a Banach space may or may not have, one that acts as a weak substitute for the parallellogram law of innerproduct norms. In particular this so called *uniform convexity* condition salvages many properties inherent in Hilbert spaces. The definition is taken from ([24], p. 59)

Definition 2.3. *Let* $(X, \|\cdot\|)$ *be a Banach space. Define the* modulus of convexity $\delta(\varepsilon)$ *for* $0 < \varepsilon \leq 2$ *by*

$$\delta(\varepsilon) := \inf\left\{1 - \frac{\|x+y\|}{2} : \|x-y\| = \varepsilon \text{ and } \|x\| = \|y\| = 1, x, y \in X\right\}.$$

The norm $\|\cdot\|$ *is said to be uniformly convex if the following holds true:* $\delta(\varepsilon) > 0$ *for every* $0 < \varepsilon \leq 2$.

Another statement of the notion of uniform convexity is that for every $\varepsilon > 0$ there corresponds some δ so that whenever $\|x\| = 1 = \|y\|$ and $\|x - y\| = \varepsilon$ one has $\|x + y\| \leq 2 - \delta$, or $\|x + y\|/2 \leq 1 - \delta/2$. Geometrically this says that no matter how close to each other two points on the boundary of the unit ball of X are, the line segment stays well inside the unit ball, in the sense that there is a uniform constant < 2 restricting the norm of the midpoint of the line segment. In particular the unit ball of a uniformly convex Banach space is strictly convex, i.e. its boundary does not contain line-segments. Examples of uniformly convex Banach spaces include $L^p(\mu), W^{k,p}(\mathbb{R}^n)$ for $1 < p < \infty$ and any Hilbert space ([10], p. 9). Here the notation $L^p(\mu)$ stands for the Banach space of p-integrable μ-measurable functions over a given measure space with given measure μ. The spaces $W^{k,p}(\mathbb{R}^n)$ will be revisited in the sequel.

Of the many properties uniformly convex spaces share with Hilbert spaces only one will be mentioned. A proof for it may be found in ([24], p. 61).

Theorem 2.4. *Let* $(X, \|\cdot\|)$ *be a uniformly convex Banach space. Then it is reflexive.*

In particular the following result will be needed.

Theorem 2.5. *The space* $L^p(\mu)$ *is uniformly convex for* $1 < p < \infty$.

An original proof can be found in [3].

Another theme which will be needed in a small way is some elementary theory of quasinormed spaces. The following definitions and theorems are taken from [1]

Definition 2.6. *A pair* $(V, \|\cdot\|)$ *is a quasinormed space if* V *is a vector space over a real or a complex scalar field and* $\|\cdot\|$ *is a* quasinorm, *i.e.*

1) $\|v\| = 0$ *if and only if* $v = 0$,

2) $\|av\| = |a|\|v\|$ *for any* $v \in V$ *and* a *from the scalar field of* V, *and finally*

3) *there is a constant* $C \geq 1$ *so that for every* $v, w \in V$ *one has* $\|v + w\| \leq C(\|v\| + \|w\|)$.

The constant C *is often referred to as the quasinorm constant of* V *(or* $\|\cdot\|$*) and* V *may also be called a* C*-quasinormed space.*

Quasinorms give rise to a topology in the same manner as normed spaces, that is the "balls" $B(x, r) = \{y \in V : \|x - y\| < r\}$ form a basis for a topology used in V. This is the topology induced by the quasinorm. One could define a concept of completeness in the spirit of the theory of normed spaces. Another (equivalent) way to introduce a notion of completeness of V is through the following result ([1], p. 59).

Theorem 2.7. *Let $(V, \|\cdot\|)$ be a C-quasinormed space. Then there is an invariant metric d on V satisfying*

$$d(x, y) \le \|x - y\|^\rho \le C d(x, y) \text{ for every } x, y \in V \tag{2.1.2}$$

where ρ is defined by $(2C)^\rho = 2$. Furthermore d induces into V its original topology.

In accordance with theorem 2.7 $(V, \|\cdot\|)$ is defined to be complete if the metric described in 2.7 is complete. Observe that any two metrics satisfying the conclusion of theorem 2.7 are equivalent. The next proposition states that, as in the case of normed spaces, completeness is characterized by the convergence of absolutely summable series.

Theorem 2.8. *A quasinormed space $(V, \|\cdot\|)$ is complete if and only if whenever $(x_n) \subset V$ is a sequence with*

$$\sum_{n=1}^{\infty} \|x_n\|^\rho < \infty$$

then the finite sums

$$s_N := \sum_{n=1}^{N} x_n$$

converge (to some element which is then denoted $\sum_{n=1}^{\infty} x_n$.)

Proof. Suppose first that V is complete and let (x_n) be a sequence satisfying the assumptions of the claim. Define $\|\cdot\|_* := d(\cdot, 0)$ given by the preceding theorem (2.7). Note that the invariance of d implies $\|x - y\|_* = d(x, y)$. The sum

$$\sum_{n=1}^{\infty} \|x_n\|_* \le \sum_{n=1}^{\infty} \|x_n\|^\rho < \infty$$

and therefore the partial sums $s_N = \sum_{n=1}^{N} x_n$ form a Cauchy sequence which then converges to $\sum_{n=1}^{\infty} x_n := s$ in the metric $\|\cdot\|_*$. Again using theorem 2.7 it can be estimated that

$$||s - \sum_{n=1}^{N} x_n|| \le C^{1/\rho} \| \sum_{n=N+1}^{\infty} x_n\|_*^{1/\rho} \overset{N\to\infty}{\longrightarrow} 0$$

which proves the "only if"-part claim.

For the opposite implication suppose that every sequence with the properties stated in the claim converges. Let (x_n) be a Cauchy sequence in the metric $\|\cdot\|_*$. Take a subsequence and relabel it (x_k) so that $\|x_k - x_{k-1}\|_* \le 2^{-k}$. Then, if (y_k) is defined by $y_1 = x_1$ and $y_k = x_k - x_{k-1}$ for $k \ge 2$, this sequence satisfies

$$\sum_{k=1}^{\infty} \|y_k\|^\rho \le C \sum_{k=1}^{\infty} \|y_k\|_* < \infty$$

so that $s_k = \sum_{j=1}^{k} y_j$ converges to some s. But $s_k = \sum_{j=1}^{k}(x_j - x_{j-1}) = x_k$ and hence the subsequence (x_k) converges to s. The Cauchy condition implies the convergence of the whole sequence and hence completeness is guaranteed. □

A particular special case of quasinormed spaces are the $L^p(\mu)$-spaces for $0 < p < 1$ and (X, μ) a given measure space. In these the quantity

$$\|f\|_p^* = \int_X |f|^p \mathrm{d}\mu = \|f\|_p^p$$

defines a metric for which theorem 2.7 holds with equality in place of the first inequality of (2.1.2). Here one can therefore take $\rho = p$.

2.2 Some theory of metric spaces

In this subsection some basic concepts of metric spaces are presented and certain connections between them are pointed out. In the extensive understanding of the theory of analysis in metric spaces these concepts play an important role. Throughout the thesis $B(x, r) = \{y \in X : d(x, y) < r\}$ will stand for the *open* ball whereas the notation $\overline{B}(x, r) = \{y \in X : d(y, x) \le r\}$ is reserved for the closed ball (unless otherwise stated). Note, in particular, that in this general context the closure $\overline{B(x, r)}$ of the open ball is not necessarily the same as $\overline{B}(x, r)$. Rather one has the one sided inclusion $\overline{B(x, r)} \subset \overline{B}(x, r)$.

Definition 2.9. *Let (X, d) be a metric space. If, for every $x \in X$ and $r > 0$, the closed ball $\overline{B}(x, r)$ is compact then (X, d) is said to be* proper.

As an immediate consequence of the previous definition it can be seen that (X, d) is proper if and only the compact sets of X are precisely those which are both closed and bounded. Moreover

Proposition 2.10. *A proper metric space (X, d) is complete and locally compact.*

Proof. Suppose (X, d) is proper. Then it is automatically locally compact. To see that it is complete take a Cauchy sequence $(x_n) \subset X$. Since it is Cauchy it is also bounded, hence there is a closed ball B so that $(x_n) \subset B$. It therefore has a convergent subsequence (x_{n_k}) with limit $x \in B$. This is actually a limit for the whole sequence (x_n): if $\varepsilon > 0$ is given and k_0 is such that $d(x, x_{n_k}) < \varepsilon/2$ whenever $k \ge k_0$ and $d(x_n, x_m) < \varepsilon/2$ whenever $n, m \ge k_0$ then for all $n \ge k_0$

$$d(x_n, x) \le d(x, x_{n_{k_0}}) + d(x_{n_{k_0}}, x_n) < \varepsilon.$$

Hence a proper space is locally compact and complete. □

The converse implication fails to hold as can be seen by taking any infinite set equipped with the discrete metric.

Definition 2.11. *A metric space (X, d) is called* doubling *with doubling constant $C_X > 0$ if, for all $r > 0$, any ball of radius $2r$ can be covered by at most C_X balls of radii r.*

Naturally there is also a connection between the doubling property and properness. Although the doubling property does not imply completeness of the space in question as can be seen by taking $(X, d) = (\mathbb{Q}, |\cdot|)$ it does imply the properness (and thus local compactness) of a complete space.

Proposition 2.12. *Let (X, d) be a complete metric space that has the doubling property defined in 2.11. Then it is proper (in particular also locally compact.)*

Proof. Let $B = \overline{B}(x, r)$ be a closed ball. Since B is complete it suffices, to demonstrate its compactness, to show B is precompact ([8], Theorem 3.5.6, p. 109). To this end let $\varepsilon > 0$. By the doubling condition B can be covered by at most C_X balls of radii $r/2$. Each of these can then be covered by at most C_X balls of radii $r/4$ which gives at most C_X^2 balls of radii $r/4$ covering B. Continuing this process k times, where k is the least integer for which $r/2^k < \varepsilon$, implies that B can be covered by at most C_X^k balls of radii $r/2^k < \varepsilon$. This completes the proof. □

In the progress of this thesis several definitions involving the above concepts will be introduced. Sometimes a certain definition will be presented in the context of complete locally compact spaces, sometimes in the context of proper metric spaces, other times in doubling proper or complete doubling spaces. Because of the interconnectedness of these concepts many of these contexts coincide. Therefore there should not be any confusion if for instance in some argument concerning a proper doubling space the completeness of that space is invoked.

Proposition 2.13. *A complete doubling space is separable.*

Proof. Let N be the doubling constant of the space and fix a point $x_0 \in X$. For each $k \in \mathbb{N}$ let $E_k = \{x_1^k, \ldots, x_{N^k}^k\}$ be the set of points, guaranteed by the doubling condition, so that

$$B(x_0, k) \subset \bigcup_{i=1}^{N^k} B(x_i^k, k/2^k).$$

Set $E = \bigcup_{k=1}^{\infty} E_k$. To prove the claim it suffices to show that E is dense.

Let $x \in X$ be fixed. If k_0 is the smallest integer so that $x \in B(x_0, k_0)$ then for every $k \geq k_0$ there is some $y_k \in E_k$ so that $d(y_k, x) \leq k/2^k$. Since $k/2^k \to 0$ as $k \to \infty$ $(y_k)_{k \geq k_0}$ is a sequence of E converging to x. □

2.3 Some measure theory

In this thesis outer measures will be used. Instead of writing the word 'outer', however, it is omitted. So it should be understood that when writing measure an outer measure is actually meant. The definitions appearing here can be found in [25].

Let (X, d) be a metric space and μ a measure on X. μ is said to be

1. a Borel-measure if all Borel sets are μ-measurable, i.e. every Borel set B satisfies
$$\mu(E) = \mu(E \setminus B) + \mu(E \cap B)$$
for all $E \subset X$.

2. a Borel regular measure if, in addition to being a Borel-measure, for every $A \subset X$ there exists a Borel set $B \supset A$ so that $\mu(A) = \mu(B)$,

3. locally finite if $0 < \mu(B) < \infty$ for every ball B.

4. doubling, if it is locally finite and if there is a constant C so that $\mu(2B) \leq C\mu(B)$ for every ball $B \subset X$. Here $2B$ denotes, as usual, the ball with same centre and two times the radius of B.

5. s-regular, $s > 0$, if there is some $b > 0$ so that $\mu(B) \geq br^s$ for any ball B with radius r and, finally

6. completely s-regular (or Alhfors s-regular) if there is a constant $c > 0$ so that $r^s/c \leq \mu(B) \leq cr^s$ for any ball B of radius r.

A few remarks might be in order: by this terminology any locally finite measure is σ-finite, and a σ-finite Borel-measure μ has the property that $C_c(X) \cap L^p(\mu)$ is dense in $L^p(\mu)$ for $1 \leq p < \infty$, provided the measure space X is locally compact ([29], p. 69). Also if μ is *any* measure on X, then the formula

$$\tilde{\mu}(A) = \inf\{\mu(B) : B \supset A,\ B \text{ Borel}\}$$

defines a Borel-regular measure on X.

The notions of s-regularity and complete s-regularity typically come across in connection to Hausdorff measures which form an important example of measures with this property. Obviously complete s-regularity is a stronger condition than s-regularity. It is also easy to see that if μ is completely s-regular, then for every ball B with radius r

$$\mu(2B) \leq c(2r)^s \leq 2^s c^2 \mu(B),$$

so that μ is also doubling. A bit less trivial is the following result, taken from [14].

Proposition 2.14. *If μ is doubling, then it is locally s-regular with $s = \log_2 C$, C being the doubling constant of μ, in the sense that if B is a ball with radius R then for every $x \in B$ and $r \leq R$*

$$\frac{\mu(B(x,r))}{\mu(B)} \geq 4^{-s} \left(\frac{r}{R}\right)^s.$$

Proof. Let B be a ball with radius R and let C be the doubling constant. If $x \in B$ and $r \leq R$, define k to be the unique integer so that $2R \leq 2^k r < 4R$. Then $\mu(B) \leq \mu(B(x, 2^k r)) \leq C^k \mu(B(x,r))$. Now

$$\frac{\mu(B(x,r))}{\mu(B)} \geq C^{-k} = 2^{-ks} \geq \left(\frac{r}{4R}\right)^s,$$

where $s = \log_2 C$. □

The doubling constant, i.e. the constant that appears in the definition of a doubling measure, is not uniquely determined but one can take an infimum of all admissible constants, thus obtaining the least possible constant. However, the above reasoning allows for any admissible constant.

Definition 2.15. *Any $s > 0$ satisfying the claim in proposition 2.14 is said to be the homogeneity exponent of μ.*

Given a general metric space X and a measure μ on X the support of μ is defined as the set

$$\{x \in X : \mu(B(x, r)) > 0 \text{ for every } r > 0\}$$

and denoted by $\operatorname{spt} \mu$. In separable metric spaces this coincides with the following definition, taken from [7]. If $\mathcal{G}$ is the collection of all closed sets $F \subset X$ for which $\mu(X \setminus F) = 0$ then

$$\operatorname{spt} \mu = \bigcap \mathcal{G}.$$

It is not difficult to see that $\operatorname{spt} \mu$ is always a closed set. If a measure μ is doubling then proposition 2.14 readily implies that $\operatorname{spt} \mu = X$.

Intuitively speaking, the doubling condition seems to indicate some sort of finite dimensional behaviour of the measure, and thus by the space which it acts upon. It is in fact true that if a metric space carries a doubling measure (i.e. some doubling measure can be defined on it) then the space itself is doubling. Although lacking in rigour this property portraits very strongly a sense of finite dimensionality of the space itself. Thus doubling measures lead naturally to a concept of dimension, conveyed quantitatively by proposition 2.14. In the sequel this will prove to be useful and much attention will be paid to metric measure spaces with a doubling measure.

Proposition 2.16. *Let X be a metric space and μ a doubling measure on X. Then X is doubling. Further the doubling constant C_X of the space can be taken to be C_μ^5, C_μ being the doubling constant of the measure.*

Proof. Let B be any ball with radius R. Denote by C_μ the doubling constant of μ and $s = \log_2 C_\mu$. Let $x_1 \in B$ be arbitrary and choose $x_2 \in B$ so that $d(x_1, x_2) > R/2$ and, in general, $x_{k+1} \in B$ so that $d(x_{k+1}, x_i) > R/2$ for all $i = 1, \dots k$. Á priori the acquired sequence may or may not be finite. However, if $N \in \mathbb{N}$ is such that $x_1, \dots, x_N$ as above *can* be chosen, the balls $B(x_i, R/4)$ are disjoint (if $x \in B(x_j, R/4) \cap B(x_i, R/4)$ for $i \neq j$ then $d(x_j, x_i) \leq d(x_j, x) + d(x_i, x) < R/2$, a contradiction) and their union is contained in $5/4B \subset 2B$. One then has the estimate

$$C_\mu \mu(B) \geq \mu(2B) \geq \sum_{i=1}^{N} \mu(B(x_i, R/4)) \geq N \frac{\mu(B)}{(4R)^s} \left(\frac{R}{4}\right)^s$$

from which a uniform upper bound is found, of

$$N \leq C_\mu \cdot 16^s = C_\mu^5$$

for the number of points it is possible to find. Now if N is the largest integer $\leq C_\mu^5$ then the $B \subset \bigcup_{i=1}^{N} B(x_i, R/2)$. This is because by the maximality of N any $x \in B$ must have the property $d(x_j, x) < R/2$ for some j, otherwise it would be the $(N+1)^{\text{th}}$ element in the constructed sequence. This proves the claim. □

As a consequence of 2.16 and the remark following 2.14 the phrase "doubling metric measure space" which á priori is ambiguous becomes a bit less so (although strictly speaking it still is). The ambiguity arises from the fact that the phrase does not specify whether the doubling property refers to that of the space or of the measure. As a consequence of 2.16 a doubling metric measure space in the second sense (i.e. with a doubling measure) implies the doubling property of the space as well. However the opposite implication fails to hold – by [30] it can be said that if the space is doubling then a doubling measure on it *exists* but it need not be the one appearing in the triplet. In the sequel the phrase might be used occasionally and it should be understood that a metric space together with a doubling measure is meant. [1]

Proposition 2.17. *Let X be a doubling metric space with doubling constant C_X and put $s = \log_2 C_X$. Then $\dim X \leq s$ where $\dim X$ denotes the Hausdorff dimension of X.*

Proof. Let B a ball or radius $R > 0$ and let $t > s$. The goal is to prove $\mathcal{H}^t(B) = 0$ or, equivalently, $\lim_{\delta\to 0} \mathcal{H}^t_\delta(B) = 0$.

If $\delta > 0$ take k to be the least integer for which $2^{-k}R < \delta$. Then the ball B can be covered with at most C_X^k balls of radii $2^{-k}R$. Now

$$\mathcal{H}^t_\delta(B) \leq \sum_{i=1}^{C_X^k} d(B_i)^t = C_X^k 2^{-kt} R^t = R^t C_X^k 2^{-sk} 2^{-k(t-s)} < R^t C_X^k C_X^{-k} (\delta/R)^{t-s}.$$

Thus

$$\mathcal{H}^t_\delta(B) \leq R^s \delta^{t-s} \to 0$$

as $\delta \to 0$. The conclusion of this is that for a fixed $t > s$, $\mathcal{H}^t(B) = 0$ for any ball B. Since $X = \bigcup_{k=1}^{\infty} kB$, where B is an arbitrary ball it follows that $\mathcal{H}^t(X) \leq \sum_k \mathcal{H}^t(kB) = 0$. Consequently $\dim X < t$ for any $t > s$, implying the claim. ☐

An immediate consequence of 2.16 and 2.17 is the following

Corollary 2.18. *If a metric space supports a doubling measure then it has finite Hausdorff dimension.*

Throughout the thesis the following notation will be used:

i) $u_A = ⨍_A u \mathrm{d}\mu = \frac{1}{\mu(A)} \int_A u \mathrm{d}\mu$ stands for the *average* of a locally integrable function u on a measure space X equipped with a measure μ.

ii) $\mathcal{M}u(x) = \sup_{r>0} |u|_{B(x,r)}$ and $\mathcal{M}_R u(x) = \sup_{0<r<R} |u|_{B(x,r)}$ denote the *Hardy-Littlewood maximal function* and its restricted version.

[1] In many cases the results stated hold regardless of the interpretation of the phrase.

One very important property of doubling measures is that for them the Lebesgue differentiation theorem is valid. The next theorem is a more general form of the Lebesgue differentiation theorem. The proof of the standard form of the Lebesgue differentiation theorem for doubling measures is similar to the case of the Lebesgue measure, see [14] and [17] where also more references can be found.

Theorem 2.19. *Let (X, d, μ) be a complete doubling metric measure space (that is, with doubling measure), let $C > 0$ be fixed and let $u \in L^1_{loc}(X)$. For μ almost every $x \in X$ the following holds true. If x_n is a sequence converging to x and $(r_n) \subset \mathbb{R}$ is a sequence converging to zero such that $r_n > Cd(x_n, x)$ for every n then*

$$u(x) = \lim_{n\to\infty} \fint_{B(x_n,r_n)} u(y)\mathrm{d}\mu(y).$$

Proof. For doubling measures the maximal function $\mathcal{M}$ has the property that there is a constant C so that if $u \in L^1(\mu)$ then

$$\mu(\{x : \mathcal{M}u(x) > t\}) \leq \frac{C||u||_{L^1}}{t} \tag{2.3.1}$$

for each $t > 0$. The proof of this can be found in [17]. Theorem 2.19 will follow easily from the more standard statement

$$\lim_{r\to 0} \fint_{B(x,r)} |u(y) - u(x)|\mathrm{d}\mu(y) = 0 \tag{2.3.2}$$

for almost every $x \in X$.

Let $K \subset X$ be any compact set and define an operator $\Lambda : L^1(K) \to \mathbb{R}$ by

$$\Lambda u(x) = \limsup_{r\to 0} \fint_{B(x,r)} |u(y) - u(x)|\mathrm{d}\mu(y)$$

For continuous u it clearly holds that $\Lambda u(x) = 0$ for every $x \in K$. Moreover Λ obeys the following estimate

$$\Lambda u(x) \leq \mathcal{M}|u - v|(x) + \Lambda v(x)$$

for any $u, v \in L^1(K)$. Given $u \in L^1_{loc}(X)$ and any $\varepsilon > 0$ there exists a continuous function $g \in L^1(K)$ so that $\|u - g\|_{L^1(K)} < \varepsilon$. Now, an estimate on the amount of $\Lambda u(x)$ which exceeds some given $t > 0$ yields

$$\begin{aligned}&\mu(\{x \in K : \Lambda u(x) > t\}) \leq \mu(\{x \in K : \mathcal{M}|u - g|(x) > t/2\})+\\&\mu(\{x \in K : \Lambda g(x) > t/2\}).\end{aligned}$$

The last term is zero since g is continuous and to estimate the first term apply (2.3.1) to get

$$\mu(\{x \in K : \mathcal{M}|u - g|(x) > t/2\}) \leq \frac{2C\|u - g\|_{L^1(K)}}{t} \leq \frac{2C\varepsilon}{t}$$

Choosing $t = \varepsilon^{1/2}$ and letting $\varepsilon \to 0$ yields

$$\Lambda u(x) = 0 \text{ for almost every } x \in K.$$

Since a complete doubling space X can be written as a countable union of compact sets 2.3.2 follows.

To see how the claim of 2.19 follows from (2.3.2) note that $B(x_n, r_n) \subset B(x, [1+C^{-1}]r_n)$ and $\mu(B(x_n, r_n)) \geq c^{-1}\mu(B(x, [1+C^{-1}]r_n))$ by 2.14. Hence

$$\limsup_{n\to\infty} \fint_{B(x_n,r_n)} |u(x) - u(y)| \mathrm{d}\mu(y) \leq \limsup_{n\to\infty} \fint_{B(x,[1+C^{-1}]r_n)} |u(x) - u(y)| \mathrm{d}\mu(y)$$

and the righthand side vanishes almost everywhere by (2.3.2).

□

A particular (and equivalent) consequence of this is that in a metric measure setting with doubling measure for a measurable set $A \subset X$ almost every point of A is a density point of A.

3 An alternative definition of the classical Sobolev space

In the classical case a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ (or $\mathbb{C}$) is said to belong to the Sobolev space $W^{k,p}(\mathbb{R}^n)$, where $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, if $f \in L^p(\mathbb{R}^n)$ and, in addition, for every multi-index α with $|\alpha| \leq k$ there is an $L^p(\mathbb{R}^n)$-function g_α such that the equation

$$\int_{\mathbb{R}^n} f(x)\partial^\alpha \phi(x)\mathrm{d}x = (-1)^{|\alpha|} \int_{\mathbb{R}^n} g_\alpha(x)\phi(x)\mathrm{d}x$$

holds for every $\phi \in C_0^\infty(\mathbb{R}^n)$. The functions g_α are more customarily denoted by $\partial^\alpha f$, and are called the αth weak derivatives of f.

The norm (or quasinorm) $||u||_{W^{1,p}(\mathbb{R}^n)} = ||u||_p + ||\nabla u||_p$ makes $W^{1,p}(\mathbb{R}^n)$ into a Banach space (or a complete quasinormed space). In the case $p > 1$ the following surprising characterization holds [14].

Theorem 3.0.1. *Let $u \in L^p(\mathbb{R}^n)$. Then $u \in W^{1,p}(\mathbb{R}^n)$ if and only if there exists an (almost) everywhere positive $g \in L^p(\mathbb{R}^n)$ and a set $N \subset \mathbb{R}^n$ of measure zero such that the inequality*

$$|u(x) - u(y)| \leq |x - y|(g(x) + g(y)) \tag{3.0.3}$$

holds for every $x, y \in \mathbb{R}^n \setminus N$.Furthermore $||\nabla u||_p \approx \inf_{g\in D(u)} ||g||_p$ where the infimum is taken over the set of the admissible functions g in the conditions of the theorem, denoted by $D(u)$.

It is said that the above inequality holds almost everywhere

Proof. To prove one direction suppose that (3.0.3) holds for some $g \in L^p(\mathbb{R}^n)$ and $N \subset \mathbb{R}^n$. Fix some $1 \leq j \leq n$. First redefine g on N to be infinity. Then the inequality (3.0.3) holds for *every* $x, y \in \mathbb{R}^n$. From Fubini's theorem it follows that for almost every line ℓ parallel to the j^{th} axis, the restriction of g to ℓ is in $L^p(\mathbb{R})$. Consequently the function u_j, defined pointwise as

$$u_j(x) = \limsup_{r\to 0} \frac{1}{r} \int_0^r \frac{u(x + te_j) - u(x)}{t} \mathrm{d}t$$

belongs to $L^p(\mathbb{R}^n)$. This is since by (3.0.3)

$$\frac{1}{r}\int_0^r \frac{|u(x+te_j)-u(x)|}{t}\mathrm{d}t \le g(x) + \frac{1}{r}\int_0^r g(x+te_j)\mathrm{d}t.$$

The measurability of u_j follows from the fact that the lim sup can be taken over the set of rationals, since for almost every x the mapping

$$r \mapsto \frac{1}{r}\int_0^r \frac{u(x+te_j)-u(x)}{t}\mathrm{d}t$$

is continuous. Passing to $\limsup r \to 0$ and using the one dimensional version of Lebesgue's differentiation theorem and the Fubini theorem one obtains

$$|u_j(x)| \le 2g(x) \tag{3.0.4}$$

for almost every $x \in \mathbb{R}^n$. Also the following lemma will be of use.

Lemma 3.0.2. *Suppose $f(x) = \limsup_{r\to 0} f_r(x)$ is measurable along with each $f_r, r > 0$ and that the mapping $r \to f_r(x)$ is continuous for almost every x. Then, for each x there is a sequence $r_k(x) \to 0$ so that*

$$f(x) = \lim_{k\to\infty} f_{r_k(x)}(x) \textit{ for almost every } x$$

and $x \mapsto f_{r_k(x)}(x)$ is measurable for all k.

Proof of lemma. For fixed $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$, let

$$s_k(x) = \sup\{0 < r < 2^{-k} : \sup_{s\le r} f_s(x) - f(x) \le 2^{-k}\}.$$

The lower semicontinuity of $g_r(x) := r \mapsto \sup_{s\le r} f_s(x)$ (for almost every fixed x) implies $0 \le \sup\limits_{s\le s_k(x)} f_s(x) - f(x) \le 2^{-k}$. Now set

$$r_k(x) = \sup\{0 < s < s_k(x) : \sup_{t\le s_k(x)} f_t(x) - f_s(x) \le 2^{-k}\}.$$

Let us prove that $r_k(x)$ satisfies the properties in the claim.

For almost every x the continuity of $r \mapsto f_r(x)$ implies that

$$f_{r_k(x)}(x) = \sup_{t\le s_k(x)} f_t(x) - 2^{-k} \tag{3.0.5}$$

and hence the estimate

$$|f(x) - f_{r_k(x)}(x)| \le |f(x) - \sup_{t\le s_k(x)} f_t(x)| + |\sup_{t\le s_k(x)} f_t(x) - f_{r_k(x)}(x)| \le 2^{-k+1}$$

shows the (almost everywhere) pointwise convergence. To prove the measurability result it suffices to demonstrate that of $x \mapsto s_k(x)$. This is since by 3.0.5 the measurability of $x \mapsto f_{r_k(x)}(x)$ follows from that of $h(x) := x \mapsto \sup_{t\le s_k(x)} f_t(x)$. Since $g_r(x)$ is lower semicontinuous in the parametre r and measurable in x the measurability of $x \mapsto s_k(x)$ will yield the measurability of h. But the measurability of $s_k(\cdot)$ (with fixed k) is evident, since for each $\alpha \in \mathbb{R}$, $\alpha \le 2^{-k}$

$$\{x : s_k(x) \ge \alpha\} = \{x : g_\alpha(x) - f(x) \le 2^{-k}\}.$$

□

This lemma enables u_j to be expressed as a limit of a sequence of functions (the dependence of the sequence on x will be omitted in the subsequent notation),

$$u_j(x) = \lim_{k\to\infty} \frac{1}{r_k}\int_0^{r_k} \frac{u(x+te_j)-u(x)}{t}\mathrm{d}t =: \lim_{k\to\infty} \frac{u_j^k(x)}{r_k}.$$

Take any $\varphi \in C_0^\infty(\mathbb{R}^n)$ and let R be such that $\operatorname{spt}\varphi \subset B(0,R)$ and $r_k \le R$ for all k. Compute

$$\int_{\mathbb{R}^n} \varphi(x)u_j(x)\mathrm{d}x = \int_{B(0,R)} \varphi(x) \lim_{k\to\infty}\frac{1}{r_k}\int_0^{r_k}\frac{u(x+te_j)-u(x)}{t}\mathrm{d}t\mathrm{d}x$$
$$= \lim_{k\to\infty}\int_{B(0,R)}\varphi(x)\frac{1}{r_k}\int_0^{r_k}\frac{u(x+te_j)-u(x)}{t}\mathrm{d}t\mathrm{d}x.$$

Here the dominated convergence theorem can be applied since

$$|u_j^k(x)| \le g(x) + \frac{1}{r_k}\int_0^{r_k} g(x+te_j)\mathrm{d}t \le g(x) + \sup_{0<r<R}\frac{1}{r}\int_0^r g(x+se_j)\mathrm{d}s$$

for almost every $x \in \mathbb{R}^n$ and the rightmost function is in $L^p(B(0,t)) \subset L^1(B(0,t))$ for any $t > 0$.

To see this estimate the integral

$$\int_{B(0,t)} \overbrace{\left(\sup_{0<r<R}\frac{1}{r}\int_0^r g(x+se_j)\mathrm{d}s\right)^p}^{:=\mathcal{M}_R^j g(x_1,\ldots,x_n)^p}\mathrm{d}x$$
$$\le \underbrace{\int_{-t}^{t}\cdots\int_{-t}^{t}}_{n-1}\left[\int_{-t}^{t}\mathcal{M}_R^j g(x_1,\ldots,x_n)^p\mathrm{d}x_j\right]\mathrm{d}x_1\cdots\mathrm{d}\hat{x}_j\cdots\mathrm{d}x_n$$
$$\le C\underbrace{\int_{-t}^{t}\cdots\int_{-t}^{t}}_{n-1}\left[\int_{-t}^{t} g(x_1,\ldots,x_n)^p\mathrm{d}x_j\right]\mathrm{d}x_1\cdots\mathrm{d}\hat{x}_j\cdots\mathrm{d}x_n$$
$$\le C\int_{\mathbb{R}^n} g^p\mathrm{d}x.$$

The middle inequality is a consequence of the boundedness of $\mathcal{M}_R^j : L^p([-t,t]) \to L^p([-t,t])$. $\mathcal{M}_R^j g$ is defined by fixing all but the jth coordinate in the argument of g and taking the restricted maximal function of the resulting function of one variable.

By changing the order of integration (both integrals are over a compact set) and making a suitable change of variables one has

$$\lim_{k\to\infty}\int_{B(0,R)}\varphi(x)\frac{1}{r_k}\int_0^{r_k}\frac{u(x+te_j)-u(x)}{t}\mathrm{d}t\mathrm{d}x$$
$$= \lim_{k\to\infty}\frac{1}{r_k}\int_0^{r_k} u(x)\int_{\mathbb{R}^n}\frac{\varphi(x-te_j)-\varphi(x)}{t}\mathrm{d}x\mathrm{d}t$$
$$= \int_{\mathbb{R}^n} u(x)\lim_{k\to\infty}\frac{1}{r_k}\int_0^{r_k}\frac{\varphi(x-te_j)-\varphi(x)}{t}\mathrm{d}t\mathrm{d}x = -\int_{\mathbb{R}^n} u(x)\partial_j\varphi(x)\mathrm{d}x.$$

Again interchanging the order of integration and limit is justified by a similar argument as before. This completes the proof of one direction of (3.0.1) since it shows that u_j is the jth weak derivative of u.

Note that from (3.0.4) the inequality $||\nabla u||_p \leq C \inf_{g \in D(u)} ||g||_p$ follows.

The other direction of the proposition is explicitly given by the pointwise inequality

$$|u(x) - u(y)| \leq C|x - y|(\mathcal{M}_{|x-y|}|\nabla u|(x) + \mathcal{M}_{|x-y|}|\nabla u|(y))$$

for almost every $x, y \in \mathbb{R}^n$ (in the sense mentioned earlier). It is enough to prove this for $C^1 \cap W^p$-functions (for which the inequality holds everywhere) since their density will then imply the general case.

Let $x \in \mathbb{R}^n$ and $R > 0$ and denote by B the ball of radius R and centre x. For any $y \in B$ one has $B \subset B(y, 2R)$. Bearing this in mind compute

$$\begin{aligned}
&|u(y) - u_B| \leq \frac{1}{|B|} \int_B |u(y) - u(z)| \mathrm{d}z \\
&= \frac{1}{|B|} \int_B \left| \int_0^1 (z - y) \cdot \nabla u(y + t(z - y)) \mathrm{d}t \right| \mathrm{d}z \\
&\leq \frac{2R}{|B|} \int_{B(x,R)} \int_0^1 |\nabla u(y + t(z - y))| \mathrm{d}t \mathrm{d}z \\
&\leq \frac{2R}{|B|} \int_0^1 \int_{B(y,2R)} |\nabla u(y + t(z - y))| \mathrm{d}z \mathrm{d}t = \frac{2R}{|B|} \int_0^1 \int_{B(0,2R)} |\nabla u(y + tz)| \mathrm{d}z \mathrm{d}t \\
&= \frac{2R}{|B|} \int_0^1 t^{-n} \int_{B(0,2tR)} |\nabla u(y + w)| \mathrm{d}w \mathrm{d}t \leq 2^{n+1} R \int_0^1 \mathcal{M}_{2R}|\nabla u|(y) \mathrm{d}t \\
&= 2^{n+1} R \mathcal{M}_{2R}|\nabla u|(y).
\end{aligned}$$

Now if $x, y \in \mathbb{R}^n$ put $z = \dfrac{x + y}{2}$ and $R = \dfrac{|x - y|}{2}$. Then $x, y \in B = B(z, R)$ and one can compute

$$|u(x) - u(y)| \leq |u(x) - u_B| + |u(y) - u_B| \leq 2^n |x - y|(\mathcal{M}_{|x-y|}|\nabla u|(x) + \mathcal{M}_{|x-y|}|\nabla u|(y)).$$

In particular $2^n \mathcal{M}|\nabla u| \in D(u)$ so

$$\inf_{g \in D(u)} ||g||_p \leq 2^n ||\mathcal{M}|\nabla u|||_p \leq C||\nabla u||_p.$$

Thus the last part of the proposition is also proven. □

This section is closed with the presentation of another, more neat proof of the "only if" part of theorem 3.0.1. What the preceding proof loses in elegance, however, it gains in that it does not at any point use the reflexivity of the spaces $L^p(\mu)$.

In fact this direction of the theorem follows quite easily from

Lemma 3.1. *Let $u \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. Then $u \in W^{1,p}(\mathbb{R}^n)$ if and only if*

$$\sup_{|h| \leq 1} \frac{\|u(\cdot + h) - u(\cdot)\|_p}{|h|} < \infty. \tag{3.0.6}$$

Proof. The necessity is a rather straightforward consequence of the fact that the jth weak partial derivative of u, since it exists as an L^p-function is almost everywhere given by the pointwise limit

$$\partial_j u(x) = \lim_{h\to\infty} \frac{u(x+he_j) - u(x)}{h}.$$

The more essential "only if" part can be seen by the following reasoning. Assuming (3.0.6) fix some standard basis vector e_j and denote, for $h \in \mathbb{R}$, $|h| \leq 1$

$$b_h^j(x) = \frac{u(x+he_j) - u(x)}{h}.$$

Then clearly for any sequence $h_k \to 0$, $|h_k| \leq 1$

$$\sup_k \|b_{h_k}^j\|_p \leq C < \infty$$

and the reflexivity of $L^p(\mathbb{R}^n)$ implies that there is a weakly convergent subsequence $b_{h_{k_m}}^j$ with limit denoted by $b^j \in L^p(\mathbb{R}^n)$. By the definition of weak convergence for any $\varphi \in C_0^\infty$ it holds that

$$\begin{aligned}\int_{\mathbb{R}^n} b^j \varphi \mathrm{d}x &= \lim_{m\to\infty} \int_{\mathbb{R}^n} b_{h_{k_m}}^j \varphi \mathrm{d}x = \lim_{m\to\infty} \int_{\mathbb{R}^n} \frac{u(x+h_{k_m}e_j) - u(x)}{h_{k_m}} \varphi(x) \mathrm{d}x \\ &= -\lim_{m\to\infty} \int_{\mathbb{R}^n} u(x) \frac{\varphi(x+h_{k_m}e_j) - \varphi(x)}{h_{k_m}} \mathrm{d}x = -\int_{\mathbb{R}^n} u \partial_j \varphi \mathrm{d}x,\end{aligned}$$

making b^j the weak jth partial derivative of u. □

Now assume (3.0.3). From this it follows that

$$\|u(\cdot + h) - u(\cdot)\|_p \leq |h| \|g(\cdot + h) + g(\cdot)\|_p \leq 2|h| \|g\|_p$$

which yields the sufficient condition (3.0.6).

4 The Hajłasz space

Proposition 3.0.1 forms the basis for this section. It's most important virtue is that it gives a characterization of the Sobolev space $W^{1,p}(\mathbb{R}^n)$ without using anything other than the metric and measure theoretic structure of $\mathbb{R}^n$. Bearing this in mind we will define a more general Sobolev space. The definition is given for an arbitrary metric measure space by which we mean the triplet (X, d, μ), where X is a set, d a metric on X and μ a locally finite Borel measure on X.

Even though in the preceding section attention was paid only to the case $p > 1$ – as indeed without this assumption (3.0.3) need not characterize the classical Sobolev space – the consideration in this section will include the whole range of positive real numbers – that is $0 < p \leq \infty$.

Definition 4.0.3. *Let (X, d, μ) be a metric measure space and $0 < p \leq \infty$. The class of Hajłasz functions, $M^{1,p}(X)$, consists of those measurable functions $u : X \to \mathbb{R}$ that belong to $L^p(X)$ and for which there exists a set $N \subset X$ and a function $g \in L^p(X)$ so that $\mu(N) = 0$, $g \geq 0$ and*

$$|u(x) - u(y)| \leq d(x,y)(g(x) + g(y)) \tag{4.0.7}$$

for every $x, y \in X \setminus N$.

The set of admissible functions g in the definition will throughout this paper be denoted by $D(u)$. Note that by defining $g = \infty$ on N the inequality (4.0.7) can be assumed to hold everywhere.

4.1 Basic properties and the density of Lipschitz functions

A very straightforward generalization of (3.0.3), definition 4.0.3 does bring the subject to a completely new, very abstract setting. Of course not very much can be said about these *Hajłasz spaces* in this level of generality. It is evident from the definition that $M^{1,p}(X) \subset L^p(\mu)$. In the spirit of the last statement of the proposition 3.0.1 a norm corresponding to the additional requirements in the definition will now be introduced, one that will make $M^{1,p}(X)$ into a Banach space or, when $0 < p < 1$, a complete quasinormed space.

Theorem 4.1.1. *The quantity*

$$||u||_{M^{1,p}(X)} = ||u||_p + \inf_{g \in D(u)} ||g||_p,$$

defines a norm on $M^{1,p}(X)$. The space $M^{1,p}(X)$ equipped with this norm is a Banach space when $p \geq 1$ and a complete quasinormed space when $0 < p < 1$.

Of course the elements of $M^{1,p}(X)$ equipped with this norm are actually equivalence classes of functions, as is usual in the L^p-theory. Hence when talking about an element $u \in M^{1,p}(X)$ it is defined only up to a set of measure zero.

Proof. Suppose c_p is the quasinorm-constant of the the quasinorm $|| \cdot ||_p$. (Obviously $|| \cdot ||_p$ is a quasinorm also when $p \geq 1$, the constant c_p being one.) Let $u, v \in W^{1,p}(X)$ and $\lambda \in \mathbb{R}$. If g_u, N_u and g_v, N_v are as in definition 4.0.3 for u and v respectively then for all $x, y \in X \setminus (N_u \cup N_v)$

$$\begin{aligned}|(u+v)(x) - (u+v)(y)| &\leq |u(x) - u(y)| + |v(x) - v(y)| \\ &\leq d(x,y)(g_u(x) + g_v(x) + g_u(y) + g_v(y))\end{aligned}$$

so

$$||u+v||_{W^{1,p}(X)} \leq ||u+v||_p + ||g_u + g_v||_p \leq c_p(||u||_p + ||v||_p + ||g_u||_p + ||g_v||_p).$$

Taking infimum over admissible g_u and g_v

$$||u+v||_{M^{1,p}(X)} \leq c_p(||u||_{M^{1,p}(X)} + ||v||_{M^{1,p}(X)})$$

is obtained. In a similar manner it is seen that

$$||\lambda u||_{M^{1,p}(X)} = |\lambda| \; ||u||_{M^{1,p}(X)}.$$

Finally $||u||_p \leq ||u||_{M^{1,p}(X)}$ so if the latter is zero, it implies that u is zero almost everywhere, hence zero as an element of $M^{1,p}(X)$. Thus $|| \cdot ||_{M^{1,p}(X)}$ is a quasinorm. As remarked when $p \geq 1$, c_p can be taken to be one, making $|| \cdot ||_{M^{1,p}(X)}$ a norm.

To prove completeness use theorem 2.8. Take a sequence $(u_k) \subset M^{1,p}(X)$ so that

$$\sum_{k=1}^{\infty} ||u_k||^{\rho}_{M^{1,p}(X)} < \infty.$$

Here $\rho = \min\{1, p\}$, see the remark following Theorem 2.8. Let $\varepsilon > 0$ and let g_k be such that $||g_k||_p \le \inf_{g \in D(u)} ||g||_p + 2^{-k}\varepsilon$ and g_k satisfies (4.0.7) with respect to u_k for some $N_k \subset X$. One has

$$\sum_{k=1}^{\infty} ||u_k||_p^\rho \le \sum_{k=1}^{\infty} ||u_k||_{M^{1,p}(X)}^\rho < \infty \text{ and}$$
$$\sum_{k=1}^{\infty} ||g_k||_p^\rho \le \sum_{k=1}^{\infty} ||u_k||_{M^{1,p}(X)}^\rho + C\varepsilon^\rho < \infty,$$

thus by the completeness of $L^p(\mu)$ there are $u, g_\varepsilon \in L^p(\mu)$ such that

$$||u - \sum_{k=1}^{n} u_k||_p \overset{n\to\infty}{\longrightarrow} 0 \text{ and } ||g_\varepsilon - \sum_{k=1}^{n} g_k||_p \overset{n\to\infty}{\longrightarrow} 0.$$

By a standard argument passing to a subsequence and relabeling the partial sums this implies the existence of sets $A, B \subset X$ of μ-measure zero so that

$$u(x) = \sum_{k=1}^{\infty} u_k(x) \quad \forall x \in X \setminus A, \text{ and} \tag{4.1.1}$$
$$g_\varepsilon(x) = \sum_{k=1}^{\infty} g_k(x) \quad \forall x \in X \setminus B. \tag{4.1.2}$$

It remains to show that the function $g^n := g_\varepsilon - \sum_{k=1}^{n} g_k$ satisfies

$$\left| u(x) - \sum_{k=1}^{n} u_k(x) - u(y) + \sum_{k=1}^{n} u_k(y) \right| \le d(x,y)(g^n(x) + g^n(y)), \quad x, y \in X \setminus N$$

for some $N \subset X$ of μ-measure zero since then it follows that

$$||u - \sum_{k=1}^{n} u_k||_{M^{1,p}(X)} \le ||u - \sum_{k=1}^{n} u_k||_p + ||g^n||_p \overset{n\to\infty}{\longrightarrow} 0.$$

Set $N = A \cup B \cup N_1 \cup N_2 \cup \cdots$. Clearly $g_n \ge 0$ and $\mu(N) = 0$. If $x, y \in X \setminus N$ then the identities (4.1.1) and (4.1.2) are available so

$$\left| u(x) - \sum_{k=1}^{n} u_k(x) - u(y) + \sum_{k=1}^{n} u_k(y) \right| = \left| \sum_{k=n+1}^{\infty} u_k(x) - \sum_{k=n+1}^{\infty} u_k(y) \right|$$
$$\le \sum_{k=n+1}^{\infty} |u_k(x) - u_k(y)| \le \sum_{k=n+1}^{\infty} d(x,y)(g_k(x) + g_k(y))$$
$$= d(x,y)(g^n(x) + g^n(y)).$$

This completes the proof. □

Examples According to Theorem 3.0.1 it holds that

$$M^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$$

with equivalent norms when $1 < p < \infty$. There are some cases, however, when $M^{1,p}(X)$ becomes "trivial" in the sense that $M^{1,p}(X) = L^p(\mu)$. One such example is given by the following.

Take $X = \mathbb{N}$ with the discrete metric and $\mu = \sum_{n \in \mathbb{N}} \delta_n$. Then μ is Borel regular and locally finite. Yet it is easy to see that if $f \in L^p(X)$ for $p > 1$ then $|f| \in D(f)$, hence $L^p(X) = M^{1,p}(X)$ with equivalent norms.

In another sense the space $M^{1,p}(X)$ is never trivial. Namely for any $0 < p < \infty$ the inclusion

$$\mathrm{LIP}_0(X) \subset M^{1,p}(X)$$

holds. Here $\mathrm{LIP}_0(X)$ denotes the set of Lipschitz-functions $X \to \mathbb{R}$ with compact support.

To see this let $u \in \mathrm{LIP}_0(X)$ and $K = \operatorname{spt} u$. Then u is μ-measurable and

$$\int_X |u|^p \mathrm{d}\mu \le \mu(K)\|u\|_\infty^p < \infty,$$

hence $u \in L^p(\mu)$. Furthermore if $g = L\chi_K$, where L is the Lipschitz constant of u then $g \in D(u)$.

The next proposition shows that for $p \in (1, \infty)$ there is a unique Hajłasz upper gradient satisfying (4.0.7) with which the infimum in the norm is obtained.

Proposition 4.1.2. *Let $u \in M^{1,p}(X)$ with $1 < p < \infty$. Then there exists a unique positive $g \in L^p(\mu)$ satisfying (4.0.7) so that*

$$||u||_{M^{1,p}(X)} = ||u||_{L^p(\mu)} + ||g||_{L^p(\mu)}.$$

Proof. Let $u \in M^{1,p}(X)$ and denote by $D(u)$ the set of admissible functions, i.e. the positive $L^p(\mu)$-functions that satisfy (4.0.7). If $g_1, \ldots, g_n \in D(u)$ and $a_1, \ldots, a_n$ are positive real numbers summing up to one then

$$\begin{aligned} |u(x) - u(y)| &= \sum_{k=1}^n a_k |u(x) - u(y)| \le \sum_{k=1}^n d(x,y)(a_k g_k(x) + a_k g_k(y)) \\ &= d(x,y) \left(\sum_{k=1}^n a_k g_k(x) + \sum_{k=1}^n a_k g_k(y) \right) \qquad \text{a.e.,} \end{aligned}$$

that is, the convex combination $\sum_{k=1}^n a_k g_k \in D(u)$. Hence $D(u)$ is a convex set, in particular it is closed if and only if it is weakly closed ([26], p. 216). But $D(u)$ is closed: if g_n is a sequence of $D(u)$ and $g_n \to g$ (in norm) then there is a subsequence that converges to g pointwise a.e. and so (4.0.7) follows for g as a pointwise limit. Now take a sequence $(g_n) \subset D(u)$ so that $||g_n||_{L^p(\mu)} \to \inf_{h \in D(u)} ||h||_{L^p(\mu)}$. Since $L^p(\mu)$ is reflexive the unit ball is compact in the weak topology. Thus take a weakly convergent subsequence of this and call the weak limit g. It follows that $g \in D(u)$ and $||g||_{L^p(\mu)} \le \lim_{n\to\infty} ||g_n||_{L^p(\mu)}$. Hence g minimizes the norm in the definition of $|| \cdot ||_{M^{1,p}(X)}$.

For the uniqueness of g the uniform convexity of $L^p(\mu)$ is used. Suppose that $g' \in D(u)$ is another function minimizing the norm, and assume that $g \neq g'$.

Then if $\|g - g'\|_{L^p(\mu)} := \varepsilon$ it follows that since $\|g\|_{L^p(\mu)} = \|g'\|_{L^p(\mu)} := a$ there is some $\delta(\varepsilon)$ for which

$$\|(g + g')/2\|_{L^p(\mu)} \leq a(1 - \delta(\varepsilon)) < a$$

which is a contradiction since $(g + g')/2 \in D(u)$ and its norm is strictly less than a. Therefore the minimizing element must be unique. □

It is evident that in the case $p = \infty$ the space $M^{1,p}(X)$ essentially consists of Lipschitz functions: if $u \in W^{1,\infty}(X)$ then there exists a positive g that is essentially bounded on $\operatorname{spt}\mu$ such that (4.0.7) is satisfied. But then it follows that

$$|u(x) - u(y)| \leq d(x, y)2||g||_\infty \quad \text{almost everywhere on } \operatorname{spt}\mu$$

which is precicely the Lipschitz condition. It follows that u has an $2||g||_\infty$-Lipschitz representative and so can be considered Lipschitz. To make this discussion a little more rigorous one has

Lemma 4.1.3. *Let $E \subset X$ and $u : E \to \mathbb{R}$ be L-Lipschitz. Then there exists a Lipschitz map $\tilde{u} : X \to \mathbb{R}$, called a Lipschitz extension of u, so that $\tilde{u}$ is L-Lipschitz and $u = \tilde{u}|_E$. Further, if $\mu(\operatorname{spt}\mu \setminus E) = 0$ then any two extensions of u agree on $\operatorname{spt}\mu$.*

Proof. The extension is given by the formula

$$\tilde{u}(x) = \inf\{Ld(x, y) + u(y) : y \in E\}, \ x \in X \tag{4.1.3}$$

and it is straightforward to verify this. To see the second part of the claim suppose its hypotheses and suppose u_1 and u_2 are two extensions of u and $x \in \operatorname{spt}\mu \setminus E$. Since $x \in \operatorname{spt}\mu$ it follows that for every $k \in \mathbb{N}$ the ball $B(x, 1/k)$ has positive measure. Since $\operatorname{spt}\mu \setminus E$ has zero measure, there must exist $x_k \in B(x, 1/k) \setminus (\operatorname{spt}\mu \setminus E) = (B(x, 1/k) \cap E) \setminus \operatorname{spt}\mu \subset B(x, 1/k) \cap E$. Now using the extension properties of u_1 and u_2 and their continuity

$$u_1(x) = \lim_{k\to\infty} u_1(x_k) = \lim_{k\to\infty} u(x_k) = \lim_{k\to\infty} u_2(x_k) = u_2(x)$$

is obtained. This proves the claim. □

Corollary 4.1.4. *If any two functions are identified whenever they agree outside a set of measure zero one has the identity*

$$M^{1,\infty}(X) = \mathrm{LIP}(X) \cap L^\infty(X).$$

Proof. Let $u \in M^{1,\infty}(X)$ and $g \in D(u)$. If $N \subset X$ is the set where $g(x) > \|g\|_\infty$ and the pointwise inequality (4.0.7) fails, then $\mu(N) = 0$. Now u is Lipschitz on $E := \operatorname{spt}\mu \setminus N$ since for every $x, y \in E$

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y)) \leq 2d(x, y)\|g\|_\infty.$$

If u_1 and u_2 are two Lipchitz extensions of $u|_E$ then $u_1(x) = u_2(x)$ for all $x \in \operatorname{spt}\mu$, according to lemma 4.1.3. In particular $u_1 = u_2$ μ-almost everywhere on X. Let $\tilde{u}$ be any Lipschitz extension of $u|_E$. Now

$$\operatorname{esssup}_{x\in X} |\tilde{u}(x)| = \operatorname{esssup}_{x\in \operatorname{spt}\mu} |\tilde{u}(x)| = \operatorname{esssup}_{x\in \operatorname{spt}\mu\setminus N} |\tilde{u}(x)| \leq \|u\|_\infty$$

Hence $\tilde{u} \in \mathrm{LIP}(X) \cap L^\infty(X)$. Moreover $\tilde{u} = u$ almost everywhere since $u = \tilde{u}$ on $\operatorname{spt}\mu \setminus N$ and $\mu(X \setminus (\operatorname{spt}\mu \setminus N)) = 0$. □

This fact makes the space $M^{1,\infty}$ a bit more concrete, something $M^{1,p}(X)$ as of yet lacks for other values of p. To account for this a sort of approximation result for $M^{1,p}(X)$ will now be proven, following [14]. The classes of infinitely differentiable or Schwartz functions are not available in metric spaces – the easiest thing is to use the class of Lipschitz functions.

Theorem 4.1.5. *Let $0 < p \leq \infty$, $u \in M^{1,p}(X)$ and $\varepsilon > 0$. Then there exists a Lipschitz function $u_\varepsilon \in M^{1,p}(X)$ so that*

$$\mu(\{x \in X : u_\varepsilon(x) \neq u(x)\}) < \varepsilon$$
$$||u - u_\varepsilon||_{M^{1,p}(X)} < \varepsilon.$$

Proof. Suppose $p = \infty$. By corollary 4.1.4 if $u \in M^{1,p}(X)$ then $u|\operatorname{spt}\mu$ can be considered Lipschitz, in the sense that there is a Lipschitz function that agrees with $u|\operatorname{spt}\mu$ almost everywhere. Therefore $u_\varepsilon = \tilde{u}$, where $\tilde{u}$ is any Lipschitz extension of $u|\operatorname{spt}\mu$, satisfies the claims of the theorem.

It can therefore be assumed that $p < \infty$. The symbol c_p again denotes the quasinorm-constant of $||\cdot||_p$. Let $\lambda > 0$. Let g and N be as in (4.0.7) and define $E_\lambda = \{x \in X \setminus N : g(x) \leq \lambda,\ |u(x)| \leq \lambda\}$.

It follows that

$$\lambda^p \mu(X \setminus E_\lambda) \leq \int_{X \setminus E_\lambda} (g^p + |u|^p)\mathrm{d}\mu \overset{\lambda \to 0}{\longrightarrow} 0$$

by the absolute continuity of $(g^p + |u|^p)\mathrm{d}\mu$ and the fact that $\mu(X \setminus E_\lambda) \to 0$. It is seen that u, restricted to E_λ, is 2λ-Lipschitz. Define

$$\tilde{u}_\lambda = \widetilde{u|_{E_\lambda}} \text{ and}$$
$$u_\lambda(x) = \begin{cases} \lambda & \text{if } \tilde{u}_\lambda(x) > \lambda, \\ \tilde{u}_\lambda(x) & \text{if } |\tilde{u}_\lambda(x)| \leq \lambda, \\ -\lambda & \text{if } \tilde{u}_\lambda(x) < -\lambda \end{cases}$$

It is easy to see that u_λ is 2λ-Lipschitz on X, $u_\lambda|_{E_\lambda} = u|_{E_\lambda}$ and further $|u_\lambda| \leq \lambda$. Hence u_λ extends $u\,|_{E_\lambda}$, defined on E_λ, to a Lipschitz function on the whole space X with the same Lipschitz constant.

Since $E_\lambda \subset \{x : u(x) = u_\lambda(x)\}$ it follows that

$$\mu(\{x : u(x) \neq u_\lambda(x)\}) \leq \mu(X \setminus E_\lambda) \tag{4.1.4}$$

which can be made arbitrarily small. One also has the estimate

$$\begin{aligned} \int_X |u - u_\lambda|^p \mathrm{d}\mu &\leq \int_{X \setminus E_\lambda} |u - u_\lambda|^p \mathrm{d}\mu \leq 2^p \left(\int_{X \setminus E_\lambda} |u|^p \mathrm{d}\mu + \int_{X \setminus E_\lambda} |u_\lambda|^p \mathrm{d}\mu \right) \\ &\leq 2^p \int_{X \setminus E_\lambda} |u|^p \mathrm{d}\mu + 2^p \lambda^p \mu(X \setminus E_\lambda) \\ &\leq 2^p \int_{X \setminus E_\lambda} |u|^p \mathrm{d}\mu + 2^p \int_{X \setminus E_\lambda} (g^p + |u|^p)\mathrm{d}\mu \overset{\lambda \to \infty}{\longrightarrow} 0. \end{aligned} \tag{4.1.5}$$

Now the proof is almost ready. To finish the argument define

$$g_\lambda(x) = \begin{cases} 0 & \text{if } x \in E_\lambda \\ g(x) + 3\lambda & \text{otherwise.} \end{cases}$$

One can calculate that for almost every $x, y \in X$

$$|u(x) - u_\lambda(x) - u(y) + u_\lambda(y)| \leq d(x,y)(g_\lambda(x) + g_\lambda(y)) :$$

If $x, y \in E_\lambda$ then this is clear since u and u_λ coincide on E_λ. If $x \in E_\lambda$ and $y \notin E_\lambda$ then $g_\lambda(x) = 0$ and $g_\lambda(y) = 3\lambda + g(y)$. Consequently

$$\begin{aligned} &|u(x) - u_\lambda(x) - u(y) + u_\lambda(y)| \leq 2\lambda d(x,y) + |u(x) - u(y)| \leq \\ &(2\lambda + g(x) + g(y))d(x,y) \leq (3\lambda + g(y))d(x,y) = d(x,y)(g_\lambda(y) + g_\lambda(x)). \end{aligned}$$

Finally if $x, y \notin E_\lambda$ then

$$|u(x) - u_\lambda(x) - u(y) + u_\lambda(y)| \leq (2\lambda + g(x) + g(y))d(x,y) \leq d(x,y)(g_\lambda(y) + g_\lambda(x)).$$

Again

$$||g_\lambda||_p^p \leq 2^p \int_{X \setminus E_\lambda} g^p \mathrm{d}\mu + 4^p \lambda^p \mu(X \setminus E_\lambda) \leq C \int_{X \setminus E_\lambda} g^p \mathrm{d}\mu \qquad (4.1.6)$$

can be made arbitrarily small. (4.1.5) and (4.1.6) together imply that $||u - u_\lambda||_{M^{1,p}(X)}$ can be made arbitrarily small and together with (4.1.4) the proof of both claims is complete. □

4.2 Embedding theorems

The elaboration of the theory of the Hajłasz spaces, as said, cannot proceed without further assumptions of the underlying space (X, d, μ). Of particular interest at this point is more intricate knowledge of the behaviour of the measure μ. To have additional structure, especially structure similar to the model case $X = \mathbb{R}^n$, one could expect the measure to be required to have similar properties with the Lebesgue measure in $\mathbb{R}^n$. In connection with the embedding theorems (in $\mathbb{R}^n$), a very significant role is played by the notion of the dimension of the underlying space. This significance comes, albeit implicitly, from the behaviour of the measure with respect to the underlying metric. Consequently a notion of dimension, or a concept relating to the quantitative behaviour of the measure, must be introduced in order to speak of embedding theorems. An obvious candidate would be the complete s-regularity of the measure since this is actually used in the proof of the classical embedding theorems. However, as it turns out a local version of the embedding theorem can be proven with the following notion of *lower s-regularity.*

Definition 4.2.1. *A measure μ on a metric measure space (X, d, μ) is said to be lower s-regular on A, for some $s \geq 0$ and $A \subset X$, if there exists a constant $b > 0$ so that*

$$\mu(B(x,r)) \geq br^s$$

for all $x \in A$ and $0 < r \leq d(A)/2$.

There is also a more technical property under which the embedding theorem can be proven with slight elaborations of the arguments presented below. For more on this see [14].

The quantity s will replace the concept of dimension. The proof of the embedding theorem is taken from [14] and [12].

Theorem 4.2.2. *Fix some number $\sigma > 1$. Let (X, μ, d) be a metric measure space whose measure μ is lower s-regular on some ball σB_0 of centre x_0 and radius r_0. Let also $0 < p \leq \infty$. Then there exist positive constants C, C' and C'' depending only on σ, s, and p so that for any $u \in M^{1,p}(X)$ and $g \in D(u)$ one of the three holds:*

1. *if $0 < p < s$ then, denoting $p^* = \dfrac{ps}{s-p}$, $u \in L^{p^*}(B_0)$ and*

$$\inf_{a\in\mathbb{R}} \left(⨍_{B_0} |u-a|^{p^*} \mathrm{d}\mu \right)^{1/p^*} \leq C \left(\frac{\mu(B_0)}{br_0^s} \right)^{1/p} r_0 \left(⨍_{\sigma B_0} g^p \mathrm{d}\mu \right)^{1/p}. \quad (4.2.1)$$

2. *If $p = s$ then*

$$⨍_{B_0} \exp\left(C' b^{1/s} \frac{|u-u_{B_0}|}{||g||_{L^p(B_0)}} \right) \mathrm{d}\mu \leq C''. \quad (4.2.2)$$

3. *If $p > s$ then $u \in L^\infty(B_0)$ and*

$$||u-u_{B_0}||_{L^\infty(B_0)} \leq C \left(\frac{\mu(B_0)}{br_0^s} \right)^{1/p} r_0 \left(⨍_{B_0} g^p \mathrm{d}\mu \right)^{1/p}. \quad (4.2.3)$$

Furthermore in the last case u is Hölder-continous in B_0 with exponent $1-s/p$: for any $x, y \in B_0$

$$|u(x) - u(y)| \leq Cb^{-1/p} ||g||_{L^p(B_0)} d(x,y)^{1-s/p}.$$

Proof. None of the claims is affected by the addition of a constant to u. This fact will be used later on. Likewise, by substituting g with $g + \left(⨍_{\sigma B_0} g^p \mathrm{d}\mu \right)^{1/p}$ it may be assumed that

$$g(x) \geq \left(⨍_{\sigma B_0} g^p \mathrm{d}\mu \right)^{1/p} > 0 \text{ for all } x \in \sigma B_0.$$

For each $k \in \mathbb{Z}$ define $E_k = \{x \in \sigma B_0 : g(x) \leq 2^k\}$. Again it is evident that u restricted to E_k is 2^{k+1}-Lipschitz (or can be modified to be) so one can also define

$$a_k = \sup_{E_k \cap B_0} |u|.$$

Finally let $r_k = (2/b)^{1/s} \mu(\sigma B_0 \setminus E_{k-1})^{1/s}$.

The sets E_k satisfy $E_k \subset E_{k+1}$ and also $\mu(\sigma B_0 \setminus E_k) \downarrow 0$ as $k \to \infty$ because for almost every $x \in \sigma B_0$ $g(x) \leq 2^k$ for some k (that is, g is finite almost everywhere). Since $2^{k-1} < g(x) \leq 2^k$ for $x \in E_k \setminus E_{k-1}$ one has

$$\sum_{k=-\infty}^{\infty} 2^{(k-1)p} \mu(E_k \backslash E_{k-1}) \leq \sum_{k=-\infty}^{\infty} \int_{E_k \backslash E_{k-1}} g^p \mathrm{d}\mu = \int_{\sigma B_0} \leq \sum_{k=-\infty}^{\infty} 2^{kp} \mu(E_k \backslash E_{k-1}),$$

in other words

$$\int_{\sigma B_0} g^p \mathrm{d}\mu \approx \sum_{k=-\infty}^{\infty} 2^{kp} \mu(E_k \setminus E_{k-1}). \quad (4.2.4)$$

Here and in the sequel the implied constants depend only on s, p and σ. The dependence on b will be stated explicitly. For r_k one has the estimate, coming from Chebychev's inequality

$$r_k \leq (2/b)^{1/s} \left(2^{-(k-1)p} \int_{\sigma B_0} g^p \mathrm{d}\mu \right)^{1/s} = (2^{p+1}/b)^{1/s} 2^{-kp/s} ||g||_p^{p/s}. \quad (4.2.5)$$

Now let $k_0 \in \mathbb{Z}$ be the least integer such that $\mu(E_{k_0-1}) < \frac{\mu(\sigma B_0)}{2} \leq \mu(E_{k_0})$. This exists since $\mu(E_k) \uparrow \mu(\sigma B_0)$ and $E_k = \varnothing$ for k such that $2^k \leq \left(⨍_{\sigma B_0} g^p \mathrm{d}\mu \right)^{1/p}$. Obviously $E_{k_0} \neq \varnothing$.

On one hand there is the estimate

$$\left(⨍_{\sigma B_0} g^p \mathrm{d}\mu \right)^{1/p} \leq g(x) \leq 2^{k_0} \text{ for } x \in E_{k_0}$$

and on the other

$$\frac{\mu(\sigma B_0)}{2} < \mu(\sigma B_0 \setminus E_{k_0-1}) \leq 2^p 2^{-pk_0} ||g||_p^p.$$

Together these lead to

$$2^{k_0} \approx \left(⨍_{\sigma B_0} g^p \mathrm{d}\mu \right)^{1/p}. \quad (4.2.6)$$

Define k_1 to be the first integer such that

$$\left(\frac{\mu(\sigma B_0)}{b r_0^s} \right)^{1/p} \max \left\{ (\sigma - 1) r_0 2^{-kp/s}, \frac{(2^{p+1}/b)^{1/s}}{1 - 2^{-p/s}} 2^{-kp/s} \right\} < (\sigma - 1) r_0 \quad (4.2.7)$$

whence $k_1 > 0$. It also follows that

$$2^{k_1} \gtrsim \left(\frac{1}{b r_0^s} \right)^{1/p}. \quad (4.2.8)$$

If $k_2 = k_0 + k_1$ then combining (4.2.6) and (4.2.8) yields

$$\left(\frac{1}{b r_0^s} \right)^{1/p} \|g\|_{L^p(\sigma B_0)} \lesssim 2^{k_2}.$$

On the other hand the minimality of k_1 together with (4.2.6) implies

$$2^{k_2} \lesssim \left(\frac{1}{b r_0^s} \right)^{1/p} \|g\|_{L^p(\sigma B_0)}.$$

Together these last two yield

$$2^{k_2} \approx \left(\frac{1}{b r_0^s} \right)^{1/p} \|g\|_{L^p(\sigma B_0)}. \quad (4.2.9)$$

Next an estimate for a_k from above is obtained. If $k > k_2$ then (4.2.7) is satisfied for k, in particular

$$\sum_{l=k_2}^{k} r_k < (\sigma - 1) r_0.$$

It can be assumed that $B_0 \cap E_k \neq \varnothing$ for if this is the case then $a_k = 0$. Take $x_k \in B_0 \cap E_k$ and consider the ball $B(x_k, r_k) \subset \sigma B_0$. The lower regularity of μ yields

$$\mu(B(x_k, r_k)) \geq b r_k^s = 2\mu(\sigma B_0 \setminus E_{k-1}),$$

implying $\mu(B(x_k, r_k) \cap E_{k-1}) = \mu(B(x_k, r_k)) - \mu(B(x_k, r_k) \setminus E_{k-1}) > 0$. In particular an element $x_{k-1} \in B(x_k, r_k) \cap E_{k-1}$ can be found. Repeating this procedure one gets a (finite) sequence $x_i \in B(x_{i+1}, r_{i+1}) \cap E_i$, $i = k_0, \ldots, k-1$. Notice that $B(x_i, r_i) \subset B(x_k, r_{k_0} + \cdots + r_k) \subset \sigma B_0$ for each $i = k_0, \ldots, k$ for all $i = k_0, \ldots, k$.

The purpose of all this was to allow for the following estimate:

$$|u(x_k)| \leq |u(x_{k_0})| + \sum_{l=k_0+1}^{k} |u(x_l) - u(x_{l-1})| \leq |u(x_{k_0})| + \sum_{l=k_0+1}^{k} 2^{l+1} d(x_l, x_{l-1})$$

$$\leq |u(x_{k_0})| + 2(2^{p+1}/b)^{1/s} ||g||_p^{p/s} \sum_{l=k_0+1}^{k} 2^{l(1-p/s)}.$$

Now use the fact mentioned in the beginning of the proof – that the claims remain unaffected if u is replaced by some $u + c$, c a constant. This allows one to assume that $\operatorname{essinf}_{E_{k_2}} |u| = 0$. Thus there is a sequence $(y_i) \subset E_{k_2}$ for which $\lim_{i\to\infty} u(y_i) = 0$. Then

$$|u(x_{k_2})| = \lim_{i\to\infty} |u(x_{k_2}) - u(y_i)| \leq 2\sigma r_0 2^{k_2+1}.$$

Taking supremum over $x_k \in E_k \cap B_0$ a final estimate

$$a_k \leq 2\left(\frac{2^{p+1}}{b}\right)^{1/s} ||g||_p^{p/s} \sum_{l=k_2+1}^{k} 2^{l(1-p/s)} + 4\sigma r_0 2^{k_2} \tag{4.2.10}$$

is obtained. This estimate holds for *all* k since if $k \leq k_2$ then $a_k \leq a_{k_2} \leq 4\sigma r_0 2^{k_2}$.

Inequalities (4.2.9) and (4.2.10) are the main ingredient in the rest of the proof. To start with (4.2.1), suppose $0 < p < s$. Then $1 - p/s > 0$ and one can estimate the sum as

$$\sum_{l=k_0+1}^{k} 2^{l(1-p/s)} \leq \frac{2^{k(1-p/s)}}{1 - 2^{p/s-1}}.$$

Use (4.2.10) to estimate

$$\int_{B_0} |u|^{p^*} \mathrm{d}\mu \leq \sum_{k=-\infty}^{\infty} a_k^{p^*} \mu(B_0 \cap (E_k \setminus E_{k-1}))$$

$$\lesssim b^{-p^*/s} ||g||_p^{pp^*/s} \sum_{k=-\infty}^{\infty} 2^{k(1-p/s)p^*} \mu(B_0 \cap (E_k \setminus E_{k-1}))$$

$$+ r_0^{p^*} 2^{k_2 p^*} \sum_{k=-\infty}^{\infty} \mu(B_0 \cap (E_k \setminus E_{k-1})).$$

The first term equals by (4.2.4)

$$b^{-p^*/s}||g||_p^{p^*p/s}\sum_{k=-\infty}^{\infty}2^{kp}\mu(B_0\cap(E_k\setminus E_{k-1}))$$
$$\le b^{-p^*p/s}||g||_p^{p(p^*/s+1)}=b^{-p^*/s}||g||^{p^*}$$

whereas the second one is, using (4.2.9)

$$2^{k_2p^*}r_0^{p*}\mu(B_0)\lesssim b^{-p^*/s}\frac{\mu(B_0)}{br_0^s}||g||_p^{p^*}.$$

Together with the lower s-regularity one has

$$\int_{B_0}|u|^{p^*}\mathrm{d}\mu\lesssim b^{p^*/s}\left(1+\frac{\mu(B_0)}{br_0^s}\right)||g||_p^{p^*}\lesssim b^{-p^*/s}\frac{\mu(B_0)}{br_0^s}||g||_p^{p^*}$$

and (4.2.1) follows.

In the second case $1-p/s=0$ so (4.2.10) becomes

$$a_k\lesssim b^{-1/s}||g||_p^{p/s}(k-k_2)+4\sigma r_0(br_0^s)^{-1/p}||g||_p$$

if $k\ge k_2$ and $a_k\le 4\sigma r_0(br_0^s)^{-1/p}||g||_p$ if $k\le k_2$. Put together,

$$a_k\lesssim b^{-1/p}||g||_p\max\{k-k_2,1\}. \tag{4.2.11}$$

Let at first C' be any positive real. To estimate the left hand side of (4.2.2) use Jensen's inequality to obtain

$$\fint_{B_0}\exp\left(C'b^{1/s}\frac{|u-u_{B_0}|}{||g||_{L^p(\sigma B_0)}}\right)\mathrm{d}\mu$$
$$\le\fint_{B_0}\exp\left(\frac{C'b^{1/s}}{||g||_p}\fint_{B_0}|u(x)-u(y)|\mathrm{d}\mu(y)\right)\mathrm{d}\mu(x)$$
$$\le\fint_{B_0}\fint_{B_0}\exp\left(C'b^{1/s}\frac{|u(x)-u(y)|}{||g||_p}\right)\mathrm{d}\mu(y)\mathrm{d}\mu(x)$$
$$\le\left[\fint_{B_0}\exp\left(C'b^{1/s}|u|/||g||_p\right)\mathrm{d}\mu\right]^2.$$

This last integral is split into two parts, one over $E_{k_2}\cap B_0$ and the other over $B_0\setminus E_{k_2}$. First, however, make a choice of C' such that $\exp(CC')=2^p$, where C is the implied constant in (4.2.11). Now

$$\fint_{B_0\cap E_{k_2}}\exp\left(C'b^{1/s}|u|/||g||_p\right)\mathrm{d}\mu\le\fint_{B_0\cap E_{k_2}}\exp\left(C'b^{1/s}a_{k_2}/||g||_p\right)\mathrm{d}\mu$$
$$\le\fint_{B_0}\exp\left(C'Cb^{1/s-1/p}||g||_p/||g||_p\right)\mathrm{d}\mu=2^p.$$

The second integral can be estimated from above by (4.2.9) and (4.2.11):

$$\frac{1}{\mu(B_0)}\sum_{k=k_2+1}^{\infty}\mu(B_0\cap(E_k\setminus E_{k-1}))\exp\left(C'b^{1/s}\frac{a_k}{||g||_p}\right)$$
$$\leq \frac{1}{\mu(B_0)}\sum_{k=-\infty}^{\infty}\mu(B_0\cap(E_k\setminus E_{k-1}))\exp\left(C'C(k-k_2+1)\right)$$
$$\leq \frac{2^p}{\mu(B_0)}2^{-pk_2}\sum_{k=-\infty}^{\infty}\mu(B_0\cap(E_k\setminus E_{k-1}))2^{kp}\leq C''\frac{br_0^s}{\mu(B_0)}||g||_p^{-p}||g||_p^p\leq C''.$$

The last inequality was again a result of the lower s-regularity of μ. Thus (4.2.2) is also proven.

Finally suppose $\infty > p > s$. Here $1-p/s<0$ and the sum in (4.2.10) can be estimated as

$$\sum_{l=k_2+1}^{k}2^{l(1-p/s)}\lesssim 2^{k_2(1-p/s)},$$

leading to a uniform estimate for a_k, using again (4.2.11)

$$\begin{aligned} a_k &\lesssim b^{-1/s}||g||_p^{p/s}2^{k_2(1-p/s)}+r_0(br_0^s)^{-1/p}||g||_p \\ &\lesssim b^{-1/s}(br_0^s)^{-1/p(1-p/s)}||g||_p^{1-p/s+p/s}+r_0(br_0^s)^{-1/p}||g||_p \\ &= b^{-1/p}||g||_p r_0^{1-s/p}. \end{aligned} \tag{4.2.12}$$

This implies, in particular, that $u\in L^\infty(B_0)$. Using $||u-u_{B_0}||_{L^\infty(B_0)}\leq 2||u||_{L^\infty(B_0)} = 2\sup_{k\in\mathbb{Z}}a_k$ and rearranging the exponents (4.2.3) is obtained. In case $p=\infty$ the result is a direct consequence of the inculsions $L^\infty\subset L^q$ for any q and the fact that $||g||_\infty=\lim_{q\to\infty}||g||_q$.

The estimate (4.2.12) is also suitable for proving the stated Hölder continuity of u. Suppose $x,y\in B_0$ and assume, at first, that $r:=d(x,y)<(\sigma-1)r_0/(2\sigma)$. Then $B:=B(x,2r)\subset\sigma B_0$ and also $\sigma B\subset\sigma B_0$. Thus the embedding theorem is valid with B_0 replaced by B and one can estimate

$$|u(x)-u(y)|\leq 2||u-u_B||_{L^\infty(B)}\leq Cb^{-1/p}r^{1-s/p}||g||_{L^p(\sigma B_0)}.$$

If, on the other hand, $d(x,y)\geq(\sigma-1)r_0/(2\sigma)$ one has

$$\begin{aligned} &|u(x)-u(y)|\leq 2||u-u_{B_0}||_{L^\infty(B_0)} \\ &\leq Cb^{-1/p}r_0^{1-s/p}||g||_p\leq Cb^{-1/p}||g||_p\left(\frac{4\sigma}{\sigma-1}\right)^{1-s/p}d(x,y)^{1-s/p}. \end{aligned}$$

All in all, the Hölder continuity of u with exponent $1-s/p$ follows and the proof of Theorem 4.2.2 is complete. □

Having proved Theorem 4.2.2 with quite weak and technical assumptions, it is desirable to narrow down the class of admissible measures and gain more structure by providing a narrower, less technical definition. From section 1 the concept of a doubling measure fits this purpose, especially since as a direct consequense of proposition 2.14 a doubling measure μ is lower s-regular on any ball σB_0, for a fixed σ, with $s=\log_2 C$ and $b=\mu(\sigma B_0)(4\sigma r_0)^{-s}$.

Therefore, if the measure of the metric measure space (X,d,μ) is assumed to be doubling Theorem 4.2.2 takes on a simpler form:

Corollary 4.2.3. *Let $0 < p \leq \infty$ and $\sigma > 1$ be fixed and denote $s = \log_2 C_d$ where C_d is the doubling constant of μ. Then there exist positive constants C, C' and C'' depeding only on σ, p and C_d so that if $B \subset X$ is any ball of radius r then for any $u \in M^{1,p}(X)$ and $g \in D(u)$ one the three following holds:*

1. *if $0 < p < s$ then, denoting $p^* = \dfrac{ps}{s-p}$, $u \in L^{p^*}(B)$ and*

$$\inf_{a \in \mathbb{R}} \left(⨍_B |u - a|^{p^*} \mathrm{d}\mu \right)^{1/p^*} \leq Cr \left(⨍_{\sigma B} g^p \mathrm{d}\mu \right)^{1/p}. \tag{4.2.13}$$

2. *If $p = s$ then*

$$⨍_B \exp\left(C' \frac{\mu(\sigma B)^{1/p}}{r} \frac{|u - u_B|}{||g||_{L^p(\sigma B)}} \right) \mathrm{d}\mu \leq C''. \tag{4.2.14}$$

3. *If $p > s$ then $u \in L^\infty(B)$ and*

$$||u - u_B||_{L^\infty(B)} \leq Cr \left(⨍_{\sigma B} g^p \mathrm{d}\mu \right)^{1/p}. \tag{4.2.15}$$

   Furthermore u is Hölder-continous in B with exponent $1 - s/p$: for any $x, y \in B$

$$|u(x) - u(y)| \leq Cr^{s/p} \left(⨍_{\sigma B} g^p \mathrm{d}\mu \right)^{1/p} d(x,y)^{1-s/p}.$$

5 The non-reflexivity of the Hajłasz space of a Cantor type fractal

The result of this section provides an example of a metric space equipped with a measure that is even completely regular (or Ahlfors-regular), the Hajłasz space for $p > 1$ of which still fails to be reflexive. This result is due to Kari Astala and Juha Rissanen, see [28] more on the non-reflexitivity of Hajłasz spaces for self similar fractals. For this section $p > 1$ will be fixed.

Let E denote the standard middle-thirds Cantor set and μ the $\log 2/\log 3$-dimensional Hausdorff measure on it. The measure μ is complelety regular on E. The nth step, E_n, in the construction of E consists of 2^n disjoint intervals $E_{n,k}$, $k = 1, \ldots, 2^n$, each with length 3^{-n} and measure $\mu(E_{n,k}) = 2^{-n}$. Define the functions u_n for $n \in \mathbb{N}$ by

$$u_n = \sum_{k=1}^{2^n} (-1)^{k+1} \chi_{E \cap E_{n,k}}.$$

By construction for each natural number n the function u_n is $2 \cdot 3^n$-Lipschitz, has absolute value 1 and integral average 0. Thus for example $||u_n||_p = 1$ and $||u_n||_{M^{1,p}(E)} \leq 1 + 3^n$.

For any $a = (a_1, a_2, \ldots) \in \ell^\infty$ define

$$u_a = \sum_{n=1}^{\infty} 3^{-n} a_n u_n.$$

The first lemma of this section states that

Lemma 5.0.4. *the map $L = a \mapsto u_a$ is a linear bounded map from ℓ^∞ to $M^{1,p}(E)$.*

Proof. The linearity of the map is of course obvious. The triangle inequality yields the estimate $||u_a||_p \leq \sum_{n=1}^{\infty} 3^{-n}|a_n| \leq C||a||_{\ell^\infty}$. To see that u_a belongs to the Hajłasz-class $M^{1,p}(E)$ take $x, y \in E$ such that $x \neq y$. Let m be first natural number for which x and y are in different intervals $E_{m,k}$ and $E_{m,k'}$. Of course necessarily then $k' = k+1$ and, more importantly, $3^{-m} \leq |x-y| \leq 3^{-m+1}$. In particular for all indices n less than m the values of u_n at x and y coincide – that is $u_n(x) = u_n(y)$ for all $n < m$. The difference is therefore

$$|u_a(x) - u_a(y)| = \left|\sum_{n=m}^{\infty} 3^{-n} a_n (u_n(x) - u_n(y))\right| \leq 2 \sum_{n=m}^{\infty} 3^{-n}|a_n|$$

$$\leq 2||a||_{\ell^\infty} \sum_{n=m}^{\infty} 3^{-n} \leq C 3^{-m} ||a||_{\ell^\infty} \leq C|x-y|||a||_{\ell^\infty}.$$

This shows that u_a is Lipschitz and thus $\inf_{g \in D(u_a)} ||g||_p \leq C||a||_{\ell^\infty}$. All in all one has $||u_a||_{M^{1,p}(E)} \leq C||a||_{\ell^\infty}$. □

The next step is to show the injectivity of L. To this purpose it is convenient to derive an expression for recovering the coefficients a_n of the bounded sequence a from the function u_a.

Lemma 5.0.5. *For a given u_a the coefficients a_n can be recovered by*

$$3^{-n} a_n = \int_E u_n u_a \mathrm{d}\mu. \tag{5.0.16}$$

In particular, L is an injective map.

Proof. By Lebesque's dominated convergence

$$\int_E u_n u_a \mathrm{d}\mu = \sum_{m=1}^{\infty} 3^{-m} a_m \int_E u_n u_m \mathrm{d}\mu.$$

In order to prove (5.0.16) it therefore suffices to show that $\int_E u_n u_m \mathrm{d}\mu = \delta_{nm}$. If $n = m$ one has $u_n u_n = |u_n|^2 = 1$ so this case is obvious. In case $n \neq m$ it can be assumed that $n < m$ since the other option ($n > m$) follows from this by interchanging the roles of n and m. But it is easy to see that

$$\int_{E \cap E_{n,k}} u_m \mathrm{d}\mu = 0$$

for any $k = 1, \ldots, 2^n$. Consequently

$$\int_E u_n u_m \mathrm{d}\mu = \sum_{k=1}^{2^n} (-1)^{k+1} \int_{E \cap E_{n,k}} u_m \mathrm{d}\mu = 0.$$

□

Let F denote the image of ℓ^∞ under L. The aim of the ongoing construction is ultimately to prove that F is isomorphic to ℓ^∞. This will imply in particular that the Hajłasz space of E is not reflexive. So far it has been established that L is an injective map from ℓ^∞ to F. The remaining part is to show that the inverse $L^{-1} : F \to \ell^\infty$ is also bounded. This is formulated explicitly by the next lemma.

Lemma 5.0.6. *The inverse of $L : \ell^\infty \to F$, given by*

$$S := u \mapsto \left(3^n \int_E u_n u \mathrm{d}\mu\right)_{n=1}^{\infty}, \tag{5.0.17}$$

is bounded.

Proof. The expression (5.0.17) is a direct consequence of the previous lemma. Albeit valid it is not a very useful tool in understanding the boundedness of S. Two slightly different expressions will be derived instead, the combination of which will yield the desired result.

Denote $A_n = E \cap \bigcup_{k=1}^{2^{n-1}} E_{n,2k-1}$ and $B_n = E \cap \bigcup_{k=0}^{2^{n-1}} E_{n,2k}$. In a few words A_n is the set E from which every even-indexed (with respect to k) interval $E_{n,k}$ of the nth step is removed and B_n likewise with odd-indexed intervals removed. Therefore $u_n = 1$ on A_n. As in the proof of 5.0.5 for $n < m$ one has

$$\int_{E \cap E_{n,k}} u_m \mathrm{d}\mu = 0$$

for all $k = 1, \ldots, 2^n$ so $\int_{A_n} u_m \mathrm{d}\mu = \sum_{k=1}^{2^{n-1}} \int_{E \cap E_{n,2k-1}} u_m \mathrm{d}\mu = 0$. Unlike in 5.0.5, the situation here is not symmetric with respect to n and m because the set the functions are integrated over depends on n. However, if $n > m$, each interval $E_{m,k}$ is divided into 2^{n-m} intervals $E_{n,k'}$ *half* of which belong to the set A_n. Then

$$\int_{A_n \cap E_{m,k}} u_m \mathrm{d}\mu = (-1)^{k+1} \mu(A_n \cap E_{m,k}) = (-1)^{k+1} \cdot 2^{n-m-1} \cdot 2^{-n} = (-1)^{k+1} 2^{-m-1},$$

yielding

$$\int_{A_n} u_m \mathrm{d}\mu = \sum_{k=1}^{2^n} 2^{-m-1} (-1)^{k+1} = 0.$$

Finally $\int_{A_n} u_n \mathrm{d}\mu = \mu(A_n) = 1/2$ so all in all

$$\int_{A_n} u_m \mathrm{d}\mu = \frac{1}{2} \delta_{nm}. \tag{5.0.18}$$

As before this identity enables S to be written in a slightly different form compared to (5.0.17). As promised one other form will be found, based on the expression

$$\int_{A_n} u_m(x + 2 \cdot 3^{-n}) \mathrm{d}\mu(x) = -\frac{1}{2} \delta_{nm}. \tag{5.0.19}$$

In fact the translation in the integrand in (5.0.19) is the reason for taking the integral over A_n instead of the whole space E. The problem with E is demonstrated by the following: if $x \in E \cap E_{n,2k-1}$ then $x + 2 \cdot 3^{-n} \in E \cap E_{n,2k}$ but if, instead, $x \in E \cap E_{n,2k}$ then $x + 2 \cdot 3^{-n} \notin E$ (thus the intervals $E_{n,2k}$ were excluded in A_n).

To prove (5.0.19) notice that $u_n(x + 2 \cdot 3^{-n}) = -1$ on A_n, giving

$$\int_{A_n} u_n(x + 2 \cdot 3^{-n}) \mathrm{d}\mu(x) = -1/2.$$

If $m > n$ then

$$\int_{E \cap E_{n,2k-1}} u_m(x + 2 \cdot 3^{-n}) \mathrm{d}\mu(x) = \int_{E \cap E_{n,2k}} u_m \mathrm{d}\mu = 0$$

for all $k = 1, \ldots, 2^{n-1}$ while in the case $n > m$ the partition of $E_{m,k}$ into 2^{n-m} separate intervals $E_{n,k'}$ can be used so that

$$\int_{A_n \cap E_{m,k}} u_m(x + 2 \cdot 3^{-n}) \mathrm{d}\mu(x) = \int_{B_n \cap E_{m,k}} u_m \mathrm{d}\mu = (-1)^{k+1} 2^{-m-1}.$$

Both these cases imply, in a similar manner than before the desired result (5.0.19).

The identities (5.0.18) and (5.0.19) easily imply, respectively, the expressions

$$\begin{aligned} -\frac{1}{2} 3^{-n} a_n &= \int_{A_n} u_a(x + 2 \cdot 3^{-n}) \mathrm{d}\mu(x) \\ \frac{1}{2} 3^{-n} a_n &= \int_{A_n} u_a \mathrm{d}\mu \end{aligned}$$

for recovering the coefficients from the function. These two combined yield the following expression for S: if $u \in F$ and $S_n(u)$ denotes the nth term of the sequance $S(u)$ then

$$S_n(u) = 3^n \int_{A_n} [u(x) - u(x + 2 \cdot 3^{-n})] \mathrm{d}\mu(x).$$

Now let $u \in F$ and $g \in D(u)$. Estimate

$$\begin{aligned} |S_n(u)| &\leq 3^n \int_{A_n} |u(x) - u(x + 2 \cdot 3^{-n})| \mathrm{d}\mu(x) \\ &\leq 2 \int_{A_n} [g(x) + g(x + 2 \cdot 3^{-n})] \mathrm{d}\mu = 2 \int_E g \mathrm{d}\mu \leq 2||g||_p. \end{aligned}$$

Thus $||S(u)||_{\ell^\infty} \leq 2||u||_{M^{1,p}(E)}$. □

The last theorem completes the proof of the fact that $\ell^\infty \cong F$. Suppose contrary to what has been already said, that the Hajłasz space $M^{1,p}(E)$ *is* reflexive. With the aid of lemmas 2.1.1 and 2.1.2 this implies that F, and thus ℓ^∞, is also reflexive. This contradiction in turn shows the ultimate result: the Hajłasz space for $p > 1$ over the Cantor set is not reflexive.

6 Newtonian spaces and the Poincaré inequality

Another possible generalization of the classical Sobolev space of exponent $p \geq 1$ to the metric measure theoretic setting is the so called Newtonian space $N^{1,p}(X)$ which utilizes the notion of *upper gradients* and p-weak upper gradients. It was first defined in [31]. This approach is more geometric and in general the space is better behaved than the Hajłasz space. Upper gradients have perhaps more to do with the geometry of the underlying space whereas Hajłasz gradients somehow merge the measure theoretic structure into the metric one so as to remove some oddities arising from the pure geometric structure of the underlying space. For instance, given any space (X, d, μ) any set of μ-measure zero can be removed from X leaving the corresponding Hajłasz space unaffected (this follows directly from definition 4.0.3). This is not the case for $N^{1,p}(X)$.

In the context of this section various assumptions about the metric measure space will be employed. These will mostly be specified in the beginning of each subsection. This exposition largely follows [14], wherein additional information and more references on the subject can be found.

6.1 Rectifiable curves

In this subsection (X, d) will – unless otherwise specified – be a separable metric space with no extra structure. μ, a measure on X, when present will be assumed to be Borel regular with respect to the metric d but no other conditions will be imposed.

Given two points $x, y \in X$, curve γ joining (or connecting) them is a continuous mapping $f : [a, b] \to X$ for which $f(a) = x$ and $f(b) = y$. The mapping f is called a parametrization of γ and is by no means unique.

Definition 6.1.1. *Two continuous mappings $f : [a, b] \to X$ and $g : [a', b'] \to X$ are said to parametrize the same curve if there exists a homeomorphism $\varphi : [a, b] \to [a', b']$ so that $g = f \circ \varphi$.*

A rectifiable curve, then, is a curve for which there exists a rectifiable parametrization, i.e. a continuous $f : [a, b] \to \gamma,\ f(a) = x,\ f(b) = y$ such that the following quantity is finite:

$$\ell(f) = \sup\{\sum_{i=1}^{n} d(f(a_i), f(a_{i-1})) : a = a_0 < \cdots < a_n = b \text{ is a partition of } [a, b]\}.$$

This is called the *length* of f. It is in fact independent of parametrization – in other words if f and g are two parametrizations of the same curve then they have the same length, $\ell(f) = \ell(g)$. Consequently the notation $\ell(\gamma)$ will be used for the length of a curve γ and no parametrization need be fixed.

ℓ can be thought of as the one dimensional Hausdorff measure with one exception. Since a parametrization can overlap itself in a set of positive $\mathcal{H}^1$ measure ℓ measures not the image of the parametrization as such (which is what $\mathcal{H}^1$ does) but rather the curve as the parametrization "sees" it. ℓ still has a property of additivity.

Lemma 6.1.2. *Lemma Let $f : [a,b] \to \gamma$ be a curve in X and $a \le t \le s \le u \le b$. Then $\ell(f|_{[t,u]}) = \ell(f|_{[t,s]}) + \ell(f|_{[s,u]})$*

Proof. If $t = a_0 < \cdots < a_n = s$ and $s = b_0 < \cdots < b_m = u$ are partitions of $[t,s]$ and $[s,u]$, respectively, then $t = a_0 < \cdots a_n < b_1 \cdots b_m = u$ forms a partition of $[t,u]$, hence

$$\ell(f|_{[t,u]}) \ge \sum_{i=1}^{n} d(f(f(a_i), f(a_{i-1})) + \sum_{j=1}^{m} d(f(b_i), f(b_{i-1}))$$

which yields $\ell(f|_{[t,u]}) \ge \ell(f|_{[t,s]}) + \ell(f|_{[s,u]})$.

The other inequality follows from the observation that any partition $t = a_0 < \cdots < a_n = u$ can be split into partitions $t = a_0 < \cdots a_m \le s$ and $s < a_{m+1} < \cdots < a_n = u$. Again this leads to

$$\sum_{i=1}^{n} d(f(a_i), f(a_{i-1})) \le \sum_{i=1}^{m} d(f(a_i), f(a_{i-1})) + d(f(s), f(a_m))$$
$$+ d(f(a_{m+1}), f(s)) + \sum_{j=m+2}^{n} d(f(a_i), f(a_{i-1})) \le \ell(f|_{[t,s]}) + \ell(f|_{[s,u]})$$

and $\ell(f|_{[t,u]}) \le \ell(f|_{[t,s]}) + \ell(f|_{[s,u]})$. □

Definition 6.1.3. *If $\gamma : [a,b] \to X$ is a curve in X the function $s_\gamma : [a,b] \to \mathbb{R}$, defined as*

$$s_\gamma(t) = \ell(\gamma|_{[a,t]}),$$

is called the length function *associated to γ.*

Theorem 6.1.4. *The length function associated to a given parametrization f of a curve is continuous. Furthermore there is a unique 1-Lipschitz parametrization $\tilde{f} : [0, \ell(\gamma)] \to \gamma$ so that $\tilde{f} \circ s_f = f$ and, in addition $\ell(\tilde{f}|_{[0,t]}) = t$ for all $0 \le t \le \ell(\gamma)$*

Proof. Let $c \in (a,b]$ and take a sequence $t_k \to c$, $t_k < c$ for all k. Since the length function is obviously non-decreasing the limit $\lim_{k\to\infty} s_f(t_k) =: \alpha$ exists and is bounded from above by $s_f(c)$. For a given $\varepsilon > 0$ and partition $a = a_0 < \cdots < a_n = c$ for which

$$s_f(c) < \varepsilon + \sum_{i=1}^{n} d(f(a_i), f(a_{i-1}))$$

let k_0 be such that $a_{n-1} < t_k < a_n = c$ whenever $k \ge k_0$. The string of estimates

$$s_f(c) < \varepsilon + \sum_{i=1}^{n-1} d(f(a_i), f(a_{i-1})) + d(f(t_k), f(a_{n-1})) + d(f(c), f(t_k))$$
$$\le \varepsilon + s_f(t_k) + d(f(c), f(t_k))$$

which holds for any $k \ge k_0$ yields,upon passing to the limit $k \to \infty$, $s_f(c) < \varepsilon + \alpha$. Since ε was arbitrary it follows that $\alpha = s_f(c)$. Similarly it can be shown that $\lim_{t\to c^-} s_f(t) = s_f(c)$ for $c \in [a,b)$ and the continuity of s_f follows.

Next define $\tilde{f}$ as $\tilde{f}(t) = f(s_f^{-1}(t)),\ t \in [0, \ell(\gamma)]$. To prove that this is well defined, suppose $x, y \in s_f^{-1}(t),\ t \in [0, \ell(\gamma)]$ and assume $x < y$. By lemma 6.1.2

$$t = s_f(y) = \ell(f|_{[a,y]}) = \ell(f|_{[a,x]}) + \ell(f|_{[x,y]}) = t + \ell(f|_{[x,y]}),$$

hence $d(f(x), f(y)) \le \ell(f|_{[x,y]}) = 0$. Clearly $\tilde{f}$ satisfies $\tilde{f} \circ s_f(t) = f(t)$ for every t in the domain of f. For $s < t \in [0, \ell(\gamma)]$ and $y \in s_f^{-1}(s), x \in s_f^{-1}(t)$ one clearly has $\tilde{f}|_{[s,t]} = f|_{[y,x]}$, in particular $\ell(\tilde{f}|_{[0,t]}) = \ell(f|_{[0,x]}) = s_f(x) = t$. This, in turn, implies that

$$\begin{aligned} &d(\tilde{f}(t), \tilde{f}(s)) \le d(\tilde{f}|_{[s,t]}) \le \ell(\tilde{f}|_{[s,t]}) = \ell(f|_{[y,x]}) = \ell(f|_{[0,x]}) - \ell(f|_{[0,y]}) \\ &= \ell(\tilde{f}|_{[0,t]}) - \ell(\tilde{f}|_{[0,s]}) = t - s. \end{aligned}$$

The last remaining task is to prove the uniqueness of $\tilde{f}$. Suppose on the contrary that g and v are two different functions for which $g \circ s_f = f = v \circ s_f$. Then, if $x = s_f(t)$,

$$d(g(t), v(t)) = d(g \circ s_f(x), v \circ s_f(x)) = d(f(x), f(x)) = 0.$$

This is possible for any t since s_f is a surjection. Thus g and v have to be the same. □

Definition 6.1.5. *Given a curve f, the unique parametrization $\tilde{f}$ as assured by theorem 6.1.4 is called the* arc length parametrization *of f.*

The next theorem shows that even different parametrizations of a curve lead to the same arc length parametrization. Thus the notation $\tilde{\gamma}$ – where no reference is made to the particular parametrization – is justified and will be used hereafter.

Proposition 6.1.6. *Suppose f and g parametrize the same curve in the sense of 6.1.1. Then $\tilde{f} = \tilde{g}$.*

Proof. Let $t \in [0, \ell(\gamma)]$ (here $\ell(\gamma)$ denotes the common value of $\ell(f)$ and $\ell(g)$) and suppose φ is a homeomorphism such that $g = f \circ \varphi$. Then

$$s_g^{-1}(t) = \varphi^{-1} s_f^{-1}(t)$$

as sets and therefore from the definition of the arc-length parametrization

$$\tilde{g}(t) = g(s_g^{-1}(t)) = g(\varphi^{-1} s_f^{-1}(t)) = f(s_f^{-1}(t)) = \tilde{f}(t).$$

This completes the proof of the claim. □

By proposition 6.1.6 it is quite obvious that the following definition – the culmination of this subsection – is, indeed, independent of any particular parametrizations.

Definition 6.1.7. *If $\gamma : [a, b] \to X$ is a given curve and $\rho : \gamma[a, b] \to [0, \infty]$ a Borel function, the integral of ρ over (or along) γ is defined to be*

$$\int_\gamma \rho = \int_0^{\ell(\gamma)} \rho(\tilde{\gamma}(t))\ \mathrm{d}t.$$

This concept of "line-integral" will have a very important role in what follows.

6.2 The modulus of a family of curves

Having defined integration along rectifiable curves it is now possible to consider, for instance, the question whether some Borel function has finite integral over *all* curves or perhaps no curves at all. However, in connection with merely measurable or Borel functions one should be able to speak about *almost all* curves – for example a measurable function belongs to $L^p(\mathbb{R})$ if and only if it is p-integrable over almost all lines parallel to the coordinate axes.

To this purpose it is convenient to introduce a measure in the family of all rectifiable curves in a given metric space, denoted $M(X)$. In other words

$$M(X) = \{\gamma : [a,b] \to X \text{ continuous } : \ell(\gamma) < \infty\}.$$

Definition 6.2.1. *Let $1 \leq p < \infty$ and $\Gamma \subset M(X)$ a set of curves in X. Denote by $F(\Gamma)$ the set of Borel functions $\rho : X \to [0,\infty]$ so that*

$$\int_\gamma \rho \geq 1 \text{ for all } \gamma \in \Gamma.$$

The p-modulus *of Γ is*

$$\mathrm{Mod}_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \left(\int_X \rho^p \mathrm{d}\mu \right)^{1/p}.$$

Note that if $\Gamma = \varnothing$ then $F(\Gamma)$ is the set of all Borel functions so that the definition above makes sense in this case as well. As promised, Mod_p turns out to be an outer measure.

Proposition 6.2.2. Mod_p *is a(n outer) measure on $M(X)$.*

Proof. For the empty set $F(\varnothing)$ consists of all Borel functions and consequently $\mathrm{Mod}_p(\varnothing) = 0$. If $\Gamma \subset \Delta$ then $F(\Delta) \subset F(\Gamma)$, hence $\mathrm{Mod}_p(\Gamma) \leq \mathrm{Mod}_p(\Delta)$.

Finally for a sequence $\Gamma_i \subset M(X)$ and an arbitrary $\varepsilon > 0$ take, for each $i \in \mathbb{N}$ $\rho_i \in F(\Gamma_i)$ so that $\left(\int_X \rho_i^p \mathrm{d}\mu \right)^{1/p} < \mathrm{Mod}_p(\Gamma_i) + \varepsilon/2^i$. If $\rho := \sum_i \rho_i$ then $\rho \in F(\bigcup_i \Gamma_i)$ which is seen as follows: for any $\gamma \in \bigcup_i \Gamma_i$

$$\int_\gamma \rho = \sum_i \int_\gamma \rho_i \geq 1$$

since γ belongs to some particular Γ_i. Then the estimate

$$\mathrm{Mod}_p(\bigcup_i \Gamma_i) \leq \left(\int_X \rho^p \mathrm{d}\mu \right)^{1/p} \leq \sum_i \left(\int_X \rho_i^p \mathrm{d}\mu \right)^{1/p} \leq \sum_i \mathrm{Mod}(\Gamma_i) + \varepsilon$$

for arbitrary ε demonstrates the countable subadditivity. □

If a property holds for all curves except for some set with p-modulus zero then this property is said to hold *p-almost everywhere*. For the null sets with respect to the p-modulus there is a nice characterization.

Proposition 6.2.3. *A set $\Gamma \subset M(X)$ has p-modulus zero if and only if there exists a non-negative Borel measurable function $\rho \in L^p(\mu)$ so that*

$$\int_\gamma \rho = \infty \text{ for every } \gamma \in \Gamma.$$

Proof. Suppose first that $\mathrm{Mod}_p(\Gamma) = 0$. Then there is a sequence $\rho_n \in F(\Gamma)$ so that $||\rho_n||_p < 2^{-n}$. The function defined by $\rho := \rho_1 + \rho_2 + \cdots$ has the desired property: it is non-negative, $||\rho||_p \leq \sum_i ||\rho_n||_p = 1$ and for any $\gamma \in \Gamma$

$$\int_\gamma \rho = \sum_n \int_\gamma \rho_n \geq \sum_n 1 = \infty.$$

For the other implication assume ρ is a positive p-integrable function for with $\displaystyle\int_\gamma \rho = \infty$ for every $\gamma \in \Gamma$. Then the same holds for ρ/n and hence $\rho/n \in F(\Gamma)$, yielding

$$\mathrm{Mod}_p(\Gamma) \leq \frac{||\rho||_p}{n} \text{ for all } n \in \mathbb{N}.$$

This implies $\mathrm{Mod}_p(\Gamma) = 0$. □

The following corollary demonstrates the typical use of the notion of the modulus.

Corollary 6.2.4. *If g is a Borel function and $g \in L^p(\mu)$ then $\int_\gamma |g|$ is finite for p-almost every curve.*

Proof. Let Γ be the set of curves for which $\int_\gamma |g| = \infty$. Then, according to proposition 6.2.3 $\mathrm{Mod}_p(\Gamma) = 0$. □

Proposition 6.2.5. *If u_n is a sequence of p-integrable Borel functions converging to a Borel function u in the p-norm, there exists a subsequence so that*

$$\int_\gamma |u_{n_k} - u| \stackrel{k\to\infty}{\longrightarrow} 0$$

for p-almost all curves γ.

Proof. Take the subsequence u_{n_k} so that $\displaystyle\int_X |u_{n_k} - u|^p \mathrm{d}\mu \leq 2^{-2pk}$ and denote by Γ_k set of curves γ for which $\displaystyle\int_\gamma |u_{n_k} - u| \geq 2^{-k}$. Then $2^k |u_{n_k} - u| \in F(\Gamma_k)$ yielding

$$\mathrm{Mod}_p(\Gamma_k) \leq \left(\int_X [2^k |u_{n_k} - u|]^p \mathrm{d}\mu \right)^{1/p} \leq 2^{k-2k} = 2^{-k}$$

If $\Gamma = \displaystyle\bigcap_n \bigcup_{i \geq n} \Gamma_i$ then for any $\gamma \notin \Gamma$ there is some n so that $\gamma \notin \displaystyle\bigcup_{i \geq n}$, i.e.

$$\int_\gamma |u_{n_i} - u| \leq 2^{-i},$$

for all $i \geq n$. Hence for any $\gamma \notin \Gamma$

$$\int_\gamma |u_{n_k} - u| \overset{k\to\infty}{\longrightarrow} 0.$$

Since $\mathrm{Mod}_p(\Gamma) \leq \mathrm{Mod}_p(\bigcup_{i\geq n} \Gamma_i) \leq 2^{1-n}$ for all n the proof of the proposition is completed. $\square$

6.3 Upper gradients

Earlier in this paper what are sometimes called the Hajłasz derivatives or Hajlasz upper gradients were introduced.[2] The following definition has as a starting point a weak form of the fundamental theorem of calculus

$$|u(x) - u(y)| \leq \int_0^1 |x-y||\nabla u(y + t(x-y))\mathrm{d}t,$$

and, as it turns out, it yields a less severe condition than the pointwise inequality (4.0.7).

Definition 6.3.1. *An upper gradient g of a locally integrable function u in a metric measure space (X, d, μ) is a non-negative Borel function such that*

$$|u(\gamma(b)) - u(\gamma(a))| \leq \int_\gamma g \tag{6.3.1}$$

for every rectifiable curve $\gamma \in M(X)$, $\gamma : [a, b] \to X$. Note that here the symbol γ is used both for the curve and its parametrization. If (6.3.1) holds for p-almost all curves γ g is said to be a p-weak upper gradient for u.

When speaking of $L^p(\mu)$-functions two functions coinciding outside a set of measure zero define the same $L^p(\mu)$-element but, unfortunately, what is an upper gradient for one may fail to be so for the other. Similarly if an upper gradient of a given function is changed on a set of measure zero then it may fail to remain an upper gradient of the same function. As it turns out no such difficulties occur with p-weak upper gradient – another demonstration of the usefulness of the concept of p-a.e..

Proposition 6.3.2. *If g is an upper gradient for a given u and g' agrees with g almost everywhere then g' is a p-weak upper gradient for u.*

Proof. To prove the first claim it is sufficient to demonstrate that $\int_\gamma g = \int_\gamma g'$ for p-almost every curve or, more generally that $\int_\gamma |g - g'| = 0$ for p-almost every

[2]The pointwise inequality in 4.0.3 constitutes the requirement for a Hajłasz upper gradient. In this terminology a HajlIsz upper gradient therefore need not be an element of $L^p(\mu)$. Functions satisfying the conditions of 4.0.3 will be referred to as an $L^p(\mu)$- Hajłasz upper gradient.

curve. For this purpose consider the set of curves Γ_k for which $\displaystyle\int_\gamma |g-g'| > 2^{-k}$. Obviously $2^k|g-g'| \in F(\Gamma_k)$, hence

$$\mathrm{Mod}_p(\Gamma_k) \leq 2^k \left(\int_X |g-g'|^p \mathrm{d}\mu\right)^{1/p} = 0.$$

If Γ denotes the set of curves for which $\int_\gamma |g-g'| \neq 0$ then $\Gamma = \bigcup_k \Gamma_k$ and thus $\mathrm{Mod}_p(\Gamma) = 0$. □

The situation is not so favourable if u is altered in a set of measure zero for neither g nor g' need then be even a p-weak upper gradient of the new function. This fact will make it necessary to talk about *versions* of functions in the theory of Newtonian spaces. However proposition 6.2.3 yields a nice approximation result relating upper and p-weak upper gradients.

Proposition 6.3.3. *If g is an upper gradient of an almost everywhere finite function u then for every $\varepsilon > 0$ there exists an upper gradient $g_\varepsilon \geq g$ for which*

$$||g_\varepsilon - g||_p \leq \varepsilon.$$

Proof. Suppose Γ is the set of curves for which $|u(\gamma(b)) - u(\gamma(a))| > \int_\gamma g$. By assumption $\mathrm{Mod}_p(\Gamma) = 0$ so, according to proposition 6.2.3 there exists a non-negative $L^p(\mu)$-function ρ such that

$$\int_\gamma \rho = \infty \text{ for every } \gamma \in \Gamma.$$

Set $g_\varepsilon = g + \dfrac{\varepsilon\rho}{||\rho||_p}$. Then clearly $g_\varepsilon \geq g$ and $||g_\varepsilon - g||_p \leq \varepsilon$. Also $|u(\gamma(b)) - u(\gamma(a))| \leq \displaystyle\int_\gamma g_\varepsilon$ for *all* $\gamma \in M(X)$ with infinity on the right-hand side for $\gamma \in \Gamma$. □

6.4 The Newtonian space

Let $1 \leq p < \infty$. A measurable function $u : X \to \mathbb{R}$ is said to belong to the class $\widetilde{N}^{1,p}(X)^{1,p}$ if there exists a non-negative $L^p(\mu)$-function g which is a p-weak upper gradient for u. Introduce in $\widetilde{N}(X)^{1,p}$ the norm-like expression

$$||u||_{1,p} = ||u||_p + \inf_{g\in G(u)} ||g||_p$$

where $G(u)$ denotes the set of all p-weak upper gradients of u. The space $\widetilde{N}(X)^{1,p}$ in itself exhibits some quite unwanted for properties as discussed just before. For instance the "norm" $||u||_{1,p}$ does not behave well under the normal equivalence relation $u \backsim v$ iff $u = v$ a.e.. To construct the Newtonian space a slightly modified equivalence relation will be defined.

Definition 6.4.1. *Define an equivalence relation between elements of $\widetilde{N}(X)^{1,p}$: $u \backsim v$ if and only if $||u-v||_{1,p} = 0$. The Newtonian space $N^{1,p}(X)$ is the quotient $\widetilde{N}(X)^{1,p}/\backsim$. The space will be equipped with the norm $||u||_{N^{1,p}(X)} := ||u||_{1,p}$.*

In particular when inquiring whether or not a given $L^p(\mu)$-function belongs to $N^{1,p}(X)$ the question really is: does there exist a *version* of u – i.e. some function v which agrees with u outside a set of measure zero – which belongs to $\widetilde{N}^{1,p}(X)$. While altering a function in a null set may result in the modified version no longer being an element of $\widetilde{N}^{1,p}(X)$ it is still true that

Proposition 6.4.2. *If $u, v \in \widetilde{N}^{1,p}(X)$ and $u = v$ almost everywhere then* $||u - v||_{1,p} = 0$.

Proof. Let N denote the set where u and v differ. Then $\mu(N) = 0$ and $g = \infty \cdot \chi_N \in L^p(\mu)$. By corollary 6.2.4 $\int_\gamma g < \infty$ for p-almost every curve. Since $u - v \in \widetilde{N}^{1,p}(X)$ there exists, by proposition 6.3.3, an upper gradient $g' \in L^p(\mu)$ of $w := u - v$. Here also $\int_\gamma g' < \infty$ for p-almost every curve. Along such curves $|w(\gamma(\beta)) - w(\gamma(\alpha))| \leq \int_\alpha^\beta g'(\tilde{\gamma}(t))\mathrm{d}t < \infty$ for every α and β in the domain of γ. Thus by the absolute continuity of the integral $w \circ \gamma$ is continuous (in fact absolutely continuous). If a curve satisfies both $\int_\gamma g < \infty$ and $\int_\gamma g' < \infty$ then the first condition implies that $|\gamma^{-1}(N)| = 0$ because $g(\gamma(t)) = \infty$ on $\gamma^{-1}(N)$. (Here $|A|$ denotes the Lebesgue measure of the set A.) Thus also $u - v = 0$ almost everywhere on γ with respect to the Lebesgue measure. By the second condition w is continuous along γ, implying that $w \circ \gamma = 0$ on the whole of $[a, b]$.

Since p-almost all curves indeed do satisfy both conditions above it follows that $g = 0$ is a p-weak upper gradient for $w = u - v$, hence $||u - v||_{1,p} = 0$. □

The following lemma relates the limit of upper gradients of a sequence of functions to the upper gradient of the limiting function.

Lemma 6.4.3. *Suppose that u_n is a sequence of functions converging to u in $L^p(\mu)$. Let g_n be p-weak upper gradients of u_n and let g be such that $g_n \to g$ in $L^p(\mu)$. Then there is a version of u so that g is a p-weak upper gradient of u.*

Proof. Proposition 6.3.3 allows one to replace the p-weak upper gradients g_k of u_k by upper gradients g'_k. For each k let g'_k be an upper gradient of u_k such that $\|g_k - g'_k\|_p \leq 2^{-k}$. Then $\|g - g'_k\|_p \to 0$ as $k \to \infty$. Take a subsequence g'_j so that $\displaystyle\int_\gamma |g'_j - g| \xrightarrow{j} 0$ for p-almost every curve and denote by Γ_0 the set of curves for which this convergence fails to hold. From this subsequence take yet another, indexed by j_k, $k \geq 1$, for which $u_{j_k} \xrightarrow{k} u$ pointwise almost everywhere. The functions u_{k_j} and u can be assumed to be everywhere finite. Define $\tilde{u} = \liminf_{k\to\infty} u_{j_k}$. Then $\tilde{u} = u$ almost everywhere. If $E \subset X$ denotes the set where $\liminf_{k\to\infty} |u_{j_k}| = \infty$ (whence $\mu(E) = 0$) let $\Gamma' = \{\gamma \in M(X) : \mathcal{L}^1(\gamma^{-1}E) > 0\}$ and $\Gamma'' = \{\gamma \in M(X) : \int_\gamma g = \infty\}$. Since $\rho = \infty\chi_E$ satisfies $\int_\gamma \rho = \infty$ for every $\gamma \in \Gamma'$ it follows from 6.2.3 that $\mathrm{Mod}_p(\Gamma') = 0$. Hence all these curve-families have p-Modulus zero and if $\Gamma = \Gamma' \cup \Gamma''$ then $\mathrm{Mod}_p(\Gamma) = 0$.

If $\gamma \notin \Gamma$, $\gamma : [a, b] \to X$, then $|\tilde{u} \circ \gamma(\alpha)| < \infty$ for $\mathcal{L}^1$- almost every $\alpha \in [a, b]$. To see that, in fact, $|\tilde{u} \circ \gamma(\alpha)| < \infty$ for *every* $\alpha \in [a, b]$ notice that for every k $u_{j_k} \circ \gamma$ is absolutely continuous,

$$|u_{j_k} \circ \gamma(\beta) - u_{j_k} \circ \gamma(\alpha)| \leq \int_\alpha^\beta g'_{j_k} \circ \tilde{\gamma}(t)\mathrm{d}t,$$

and moreover

$$\int_\alpha^\beta |g'_{j_k} \circ \tilde{\gamma}(t) - g \circ \tilde{\gamma}(t)| \mathrm{d}t \le \int_\gamma |g'_{j_k} - g| \overset{k\to\infty}{\longrightarrow} 0$$

for every $a \le \alpha \le \beta \le b$. Now if $\alpha \in [a,b]$ is arbitrary, estimate for every k

$$|u_{j_k} \circ \gamma(\alpha)| \le |u_{j_k} \circ \gamma(\beta) - u_{j_k} \circ \gamma(\alpha)| + |u_{j_k} \circ \gamma(\beta)| \le \int_\alpha^\beta g'_{j_k} \circ \tilde{\gamma}(t) \mathrm{d}t + |u_{j_k} \circ \gamma(\beta)|.$$

Since $\gamma^{-1}E \subset [a,b]$ has zero $\mathcal{L}^1$-measure there must exist some $\beta \in [a,b] \setminus \gamma^{-1}E$, that is, $\liminf_{k\to\infty} |u_{j_k} \circ \gamma(\beta)| < \infty$, and therefore

$$\begin{aligned}
&|\tilde{u} \circ \gamma(\alpha)| \le \liminf_{k\to\infty} |u_{j_k} \circ \gamma(\alpha)| \le \liminf_{k\to\infty} \int_\alpha^\beta g'_{j_k} \circ \tilde{\gamma}(t) \mathrm{d}t + \liminf_{k\to\infty} |u_{j_k} \circ \gamma(\beta)| \\
&\le \int_\gamma g + \liminf_{k\to\infty} |u_{j_k} \circ \gamma(\beta)| < \infty
\end{aligned}$$

Therefore, if $\gamma \notin \Gamma$ then $\tilde{u} \circ \gamma < \infty$ on $[a,b]$ and

$$\begin{aligned}
&|\tilde{u}(\gamma(b)) - \tilde{u}(\gamma(a))| = \lim_{k\to\infty} |\inf_{n\ge k} u_{j_n}(\gamma(b)) - \inf_{n\ge k} u_{j_n}(\gamma(a))| \\
&\le \limsup_{k\to\infty} |u_{j_k}(\gamma(b)) - u_{j_k}(\gamma(a))| \le \limsup_{k\to\infty} \int_\gamma g'_{j_k} = \int_\gamma g.
\end{aligned}$$

□

These preliminaries allow one to formulate a theorem stating that

Theorem 6.4.4. *The space $N^{1,p}(X)$ is a Banach space for $1 \le p < \infty$.*

Proof. That $||\cdot||_{N^{1,p}(X)}$ is a norm is obvious. As with the Hajlasz spaces completeness will be proven via the convergence of absolutely summable series.

Let then u_n be a sequence in $N^{1,p}(X)$ so that $\sum_n ||u_n||_{N^{1,p}(X)} < \infty$. For each n and given $\varepsilon > 0$ let $g_n \in G(u_n)$ and $||g_n||_p < \inf_{g\in G(u_n)} ||g||_p + \varepsilon/2^n$. Since both

$$\begin{aligned}
&\sum_n ||u_n||_p < \infty \text{ and} \\
&\sum_n ||g_n||_p < \varepsilon + \sum_n \inf_{g\in G(u_n)} ||g||_p < \infty
\end{aligned}$$

the following convergences hold in $L^p(\mu)$ (by completeness of $L^p(\mu)$):

$$\begin{aligned}
v_k &:= \sum_n^k u_n \overset{k\to\infty}{\longrightarrow} \sum_n u_n := v, \\
g_\varepsilon^k &:= \sum_n^k g_n \overset{k\to\infty}{\longrightarrow} \sum_n g_n := g_\varepsilon.
\end{aligned}$$

Clearly g_ε^k is a p-weak upper gradient of v_k, hence by lemma 6.4.3 g_ε is a p-weak upper gradient of a version of v and thus $v \in \widetilde{N}^{1,p}(X)$. It remains to show that $v_k \to v$ in $N^{1,p}(X)$. To this purpose note that $g_\varepsilon - g_\varepsilon^k$ is also a p-weak upper gradient of $v - v_k$ and therefore

$$||v-v_k||_{N^{1,p}(X)} \leq ||v-v_k||_p + ||g_\varepsilon - g_\varepsilon^k||_p \leq \sum_{n=k+1}^{\infty} ||u_n||_p + \sum_{n=k+1}^{\infty} \inf_{g \in G(u_n)} ||g||_p + \varepsilon.$$

The sums on the left-hand side converge to zero while ε is arbitrary and hence $v_k \to v$ in $N^{1,p}(X)$. □

In general $N^{1,p}(X)$ does not need to coincide with $M^{1,p}(X)$. If the underlying space X doesn't contain any rectifiable curves then self-evidently any non-negative Borel function – in particular the zero function – is an upper gradient for any $L^p(\mu)$-function u. Thus $N^{1,p}(X) = L^p(\mu)$ for such spaces (with equal norms). For instance the cantor set E encountered earlier illustrates one such example. But, as was seen, the Hajłasz space $M^{1,p}(X)$ is still not "trivial". However the spaces $N^{1,p}(X)$ and $M^{1,p}(X)$ are not completely without relation. In this spirit it shall shortly be proven that the Hajłasz space continuously embeds into the Newtonian one. Roughly speaking, the richer the structure of the curves of the underlying space the closer the resemblance between $N^{1,p}(X)$ and $M^{1,p}(X)$.

Theorem 6.4.5. *$M^{1,p}(X)$ embeds into $N^{1,p}(X)$ continuously. More specifically $M^{1,p}(X) \subset N^{1,p}(X)$ and*

$$||u||_{N^{1,p}(X)} \leq 2||u||_{M^{1,p}(X)} \text{ for all } u \in M^{1,p}(X).$$

Proof. Let $u \in M^{1,p}(X)$ be a Lipschitz map with Lipschitz constant C. Then for any curve γ and its arc-legth parametrization $\tilde{\gamma} : [0, L] \to \gamma$ the composition $u \circ \tilde{\gamma} : [0, L] \to \mathbb{R}$ is Lipschitz and hence almost everywhere differentiable. In particular

$$u(\tilde{\gamma}(a)) - u(\tilde{\gamma}(b)) = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} u \circ \tilde{\gamma}(t) \mathrm{d}t, \ 0 \leq a \leq b \leq L.$$

Let then g is a Hajłasz upper gradient of u. By redefining g to be, say, C in a set of μ-measure zero it can be assumed that the inequality

$$|u(x) - u(y)| \leq d(x, y)[g(x) + g(y)]$$

holds for *every* $x, y \in X$ and that g is a Borel function. For any $n \in \mathbb{N}$ Luzin's theorem implies the existence of an open set $U_n \subset [0, L]$ for which $|U_n| < 1/n$ and $g \circ \tilde{\gamma}$ is continuous in $[0, L] \setminus U_n =: C_n$. The sets U_n can, of course, be chosen so as to form a decreasing sequence $U_1 \supset U_2 \supset U_3 \ldots$. Let B be the set of density points of $\bigcup_{n=1}^{\infty} C_n$. Evidently $|[0, L] \setminus B| = 0$ since $|[0, L] \setminus B| \leq |U_n| < 1/n$ for every n. If $t \in B$ then

$$\frac{|\bigcup_{n=1}^{\infty} [t - 1/k, t + 1/k] \cap C_n|}{2/k} \overset{k \to \infty}{\longrightarrow} 1,$$

in particular there is for each k (sufficiently large) some $s_k \in [-1/k, 1/k] \setminus \{0\}$ so that $t + s_k \in \bigcup_{n=1}^{\infty} [t - 1/k, t + 1/k] \cap C_n$. It then follows that $t + s_k \to t$ and $g \circ \tilde{\gamma}(t + s_k) \to g \circ \tilde{\gamma}(t)$.

Letting A denote the intersection of B with poits of differentiability of $u \circ \tilde{\gamma}$ can be seen that for any $t \in A$

$$\begin{aligned}
&\left| \frac{\mathrm{d}}{\mathrm{d}t} u \circ \tilde{\gamma}(t) \right| = \lim_{k \to \infty} \frac{|u \circ \tilde{\gamma}(t + s_k) - u \circ \tilde{\gamma}(t)|}{|s_k|} \\
&\leq \lim_{k \to \infty} \frac{d(\tilde{\gamma}(t + s_k), \tilde{\gamma}(t))}{|s_k|} [g(\tilde{\gamma}(t + s_k)) + g(\tilde{\gamma}(t))] \\
&\leq \lim_{k \to \infty} [g(\tilde{\gamma}(t + s_k)) + g(\tilde{\gamma}(t))] = 2g(\tilde{\gamma}(t)).
\end{aligned}$$

Consequently

$$|u \circ \tilde{\gamma}(b) - u \circ \tilde{\gamma}(a)| \leq \int_a^b \left| \frac{\mathrm{d}}{\mathrm{d}t} u \circ \tilde{\gamma}(t) \right| \mathrm{d}t \leq \int_a^b 2g(\tilde{\gamma}(t)) \mathrm{d}t = \int_\gamma 2g.$$

Having thus demonstrated that $2g$ is an upper gradient of u whenever u is Lipschitz, take any $u \in M^{1,p}(X)$ and $u_n \in M^{1,p}(X)$ a sequence of Lipschitz functions converging to u in $M^{1,p}(X)$-norm. If g_n are Hajłasz upper gradients of u_n, g a Hajłasz upper gradient of u and $||u_n - u||_p \to 0$ and $||g_n - g||_p \to 0$ then, according to lemma 6.4.3 there is a version of g (which can be assumed to be Borel) which is a p-weak upper gradient of a version of u. Therefore

$$\inf_{g \in D(u)} ||g||_p \leq 2 \inf_{g \in G(u)} ||g||_p$$

and the theorem is proven. □

7 Spaces supporting a Poincaré inequality

Both the Newtonian and the Hałjasz spaces can be defined over any metric measure space (X, d, μ). However if X is too "thin" – like the Cantor set – then $N^{1,p}(X)$ fails to take into account any differentiability properties of its elements. The Hajłasz space $M^{1,p}(X)$ on the other hand then may fail to be reflexive, see section 3. This section is devoted to the study of a class of metric measure spaces with a rich enough structure of curves, so that one can hope to avoid the phenomena described above. Again [14] is used as a reference, unless another reference is specified.

7.1 The Hajłasz space through Poincaré inequalities

If (X, d, μ) is any metric measure space and $p \geq 1$, $u \in M^{1,p}(X)$ and $g \in D(u)$ then integrating inequality (4.0.7) in the definition of the Hajłasz space over a ball $B = B(x, r)$ yields

$$|u(x) - u_B| \leq r(g(x) + g_B)$$

and, upon second integration using Jensen's inequality

$$⨍_B |u - u_B| \mathrm{d}\mu \leq 2r \left(⨍_B g^p \mathrm{d}\mu \right)^{1/p}.$$

This is called the (strong) p-Poincaré inequality for the pair (u, g). Thus any pair (u, g) of a $M^{1,p}(X)$-function and its Hajłasz upper gradient g satisfies the p-Poincaré inequality. In fact,

Theorem 7.1.1. *If (X, d, μ) is a metric measure space with a doubling measure and $p > s/(s+1)$,s being the homogeneity exponent of μ (see 2.15), then $u \in M^{1,p}(X)$ if and only if there exists a non-negative $g \in L^p(\mu)$ for which*

$$\fint_B |u - u_B| \mathrm{d}\mu \leq Cr \left(\fint_{\sigma B} g^q \mathrm{d}\mu \right)^{1/q} \tag{7.1.1}$$

for every ball B, where r is the radius of B, for some $s/(s+1) \leq q < p$ and for some constants $C > 0$ and $\sigma \geq 1$ depending only on p and the doubling constant of μ.

Proof. Suppose that $u \in M^{1,p}(X)$, B is a ball of radius r and $g \in L^p(\mu)$ is a Hajłasz upper gradient for u. Then $u \in W^{1,s/(s+1)}(B)$ and since $s/(s+1) < s$ inequality (4.2.13) from the embedding theorem 4.2.3 is applicable, with $(s/(s+1))^* = 1$. It yields

$$\fint_B |u - u_B| \mathrm{d}\mu \leq Cr \left(\fint_{\sigma B} g^{s/(s+1)} \mathrm{d}\mu \right)^{(s+1)/s} .$$

This proves the "only if"-part of the claim.

Conversely, suppose $0 \leq g \in L^p(\mu)$ satisfies (7.1.1) for fixed constants C, σ and some $s/(s+1) \leq q < p$. Since μ is doubling the Lebesgue's differentiation theorem holds. If x and y are Lebesgue points of u denote $r_i = 2^{-i} d(x, y)$ and $B_i = B(x, r_i)$. Then the equality

$$u_{B_i} - u_{B_0(x)} = \sum_{k=0}^{i-1} [u_{B_k} - u_{B_{k+1}}]$$

leads to

$$|u(x) - u_{B_0(x)}| \leq \sum_{k=0}^{\infty} |u_{B_k} - u_{B_{k+1}}|$$

upon passing to the limit $i \to \infty$. Here B_0 is written as $B_0(x)$ to emphasize the fact that it has centre x. On the other hand

$$\begin{aligned} |u_{B_k} - u_{B_{k+1}}| &= \fint_{B_{k+1}} |u_{B_k} - u_{B_{k+1}}| \mathrm{d}\mu \leq \fint_{B_{k+1}} |u - u_{B_k}| \mathrm{d}\mu \\ &+ \fint_{B_{k+1}} |u - u_{B_{k+1}}| \mathrm{d}\mu \leq C r_{k+1} \left(\fint_{\sigma B_{k+1}} g^q \mathrm{d}\mu \right)^{1/q} \\ &+ C r_k \left(\fint_{\sigma B_k} g^q \mathrm{d}\mu \right)^{1/q} \leq C r_k \left(\mathcal{M} g^q(x) \right)^{1/q} = C 2^{-k} d(x, y) \left(\mathcal{M} g^q(x) \right)^{1/q} . \end{aligned}$$

Combining these two inequalities leads to

$$|u(x) - u_{B_0(x)}| \leq C \left(\mathcal{M} g^q(x) \right)^{1/q} \sum_{k=0}^{\infty} 2^{-k} d(x, y) = C d(x, y) \left(\mathcal{M} g^q(x) \right)^{1/q} .$$

A similar argument gives the same result with x replaced by y. Since $|u(x) - u(y)| \leq |u(x) - u_{B_0(x)}| + |u(y) - u_{B_0(y)}| + |u_{B_0(y)} - u_{B_0(x)}|$ the pointwise estimate

$$|u(x) - u(y)| \leq Cd(x,y)[(\mathcal{M}g^q(x))^{1/q} + (\mathcal{M}g^q(y))^{1/q}]$$

will be established as soon as the last term in the right-hand side of the triangle inequality-estimate is handled. For it the triangle inequality yields

$$|u_{B_0(y)} - u_{B_0(x)}| \leq |u_{B_0(x)} - u_{2B_0(x)}| + |u_{B_0(y)} - u_{2B_0(x)}|$$

and, for each term separately

$$\begin{aligned}|u_{B_0(y)} - u_{2B_0(x)}| &\leq \fint_{B_0(y)} |u - u_{2B_0(x)}| \mathrm{d}\mu \leq \frac{1}{\mu(B_0(y))} \int_{2B_0(x)} |u - u_{2B_0(x)}| \mathrm{d}\mu \\ &\leq \frac{C_\mu^2}{\mu(2B_0(x))} \int_{2B_0(x)} |u - u_{2B_0(x)}| \mathrm{d}\mu \leq CC_\mu^2 d(x,y) \left(\mathcal{M}g^q(x)\right)^{1/q}.\end{aligned}$$

(The term $|u_{B_0(x)} - u_{2B_0(x)}|$ is estimated in the same way.) These inequalities complete the proof of the claim, since

$$\| \left(\mathcal{M}g^q\right)^{1/q} \|_p^p \leq C \|g^q\|_{p/q}^{q/p} = C\|g\|_p^p$$

□

In fact as can easily be seen in the above proof the maximal functions $\mathcal{M}$ can be replaced by $\mathcal{M}_{2\sigma d(x,y)}$ because $2\sigma d(x,y)$ is the radius of the largest ball anything is integrated over. Also, the proof of the "if" part of theorem 7.1.1 goes through even with $q = p$. Hence a more careful study of this proof yields

Corollary 7.1.2. *Suppose the pair (u,g) satisfies the p-Poincaré inequality, $p \in (s/(s+1), \infty)$*

$$\fint_B |u - u_B| \mathrm{d}\mu \leq Cr \left(\fint_{\sigma B} g^p \mathrm{d}\mu \right)^{1/p}$$

for any ball $B = B(x,r)$. Then

$$|u(x) - u(y)| \leq Cd(x,y)[\left(\mathcal{M}_{2\sigma d(x,y)} g^p(x)\right)^{1/p} + \left(\mathcal{M}_{2\sigma d(x,y)} g^p(y)\right)^{1/p}]$$

μ-almost everywhere.

7.2 Lip and lip

Recall that a function $F : (X,d) \to (Y,d')$ between two metric spaces (X,d) and (Y,d') is called *Lipschitz* if there is a constant $0 < L < \infty$ so that

$$d'(f(x), f(y)) \leq Ld(x,y) \text{ for every } x, y \in X.$$

The collection of all Lipschitz-functions $X \to \mathbb{R}$ is denoted as $\mathrm{LIP}(X)$ and the Lipschitz constant of $u \in \mathrm{LIP}(X)$ is $\mathrm{LIP}(u)$ (unless some other symbol is specified). Given any $u \in \mathrm{LIP}(X)$ the Lip and lip of u are functions $X \to \mathbb{R}$ defined everywhere as follows.

Definition 7.2.1. *Let $u \in \mathrm{LIP}(X)$. Then for every $x \in X$*

1.
$$\operatorname{Lip} u(x) = \limsup_{r \to 0} \sup_{d(x,y)<r} \frac{|u(x) - u(y)|}{r}$$

2.
$$\operatorname{lip} u(x) = \liminf_{r \to 0} \sup_{d(x,y)<r} \frac{|u(x) - u(y)|}{r}.$$

Proposition 7.2.2. *For every $r > 0$ and Lipschitz-function u the function*

$$L_r u(x) = \sup_{d(x,y)<r} \frac{|u(x) - u(y)|}{r}$$

is lower semicontinuous.

Proof. It suffices to show that for every $t \in \mathbb{R}$ the set $A_t = \{x \in X : u(x) > t\}$ is open. Of course t can be assumed to be non-negative. Let $x \in A_t$, that is,

$$\sup_{d(x,y)<r} \frac{|u(x) - u(y)|}{r} > t.$$

Thus there is some $y' \in B(x, r)$ so that $|u(x) - u(y')| > rt$. Choose an $\varepsilon > 0$ for which $|u(x) - u(y')| > rt + \varepsilon$. On the other hand by the continuity of u there exists some $\delta \in (0, r - d(x, y')]$ so that

$$|u(x) - u(y)| < \varepsilon \text{ for every } y \in B(x, \delta).$$

Now if $y \in B(x, \delta)$ then $d(y, y') \leq d(y, x) + d(x, y') < \delta + d(x, y') < r - d(x, y') + d(x, y') = r$ and hence

$$|u(y) - u(y')| \geq |u(y') - u(x)| - |u(x) - u(y)| > rt + \varepsilon - \varepsilon = rt.$$

Thus $L_r u(y) > t$ for each $y \in B(x, \delta)$ which implies that A_t is open. □

Corollary 7.2.3. *Given any $u \in \mathrm{LIP}(X)$ the functions* $\operatorname{Lip} u$ *and* $\operatorname{lip} u$ *are Borel functions.*

Proof. This follows directly from 7.2.2 since Lip and lip can be expressed as

$$\operatorname{Lip} u(x) = \limsup_{\substack{r \to 0 \\ r \in \mathbb{Q}}} L_r u(x)$$

and

$$\operatorname{lip} u(x) = \liminf_{\substack{r \to 0 \\ r \in \mathbb{Q}}} L_r u(x).$$

To see this fix some $x \in X$. In the first case note that, since $u_t(x) := t \mapsto \sup_{r<t} \sup_{y \in B(x,r)} \frac{|u(x) - u(y)|}{r}$ is increasing, one has

$$\operatorname{Lip} u(x) = \inf_{t>0} u_t(x)$$

and similarly

$$\limsup_{\substack{r\to 0\\ r\in\mathbb{Q}}} L_r u(x) = \inf_{t\in\mathbb{Q}^+} \sup_{\substack{r<t\\ r\in\mathbb{Q}}} \sup_{y\in B(x,r)} \frac{|u(x)-u(y)|}{r}.$$

But in fact

$$\sup_{\substack{r<t\\ r\in\mathbb{Q}}} \sup_{y\in B(x,r)} \frac{|u(x)-u(y)|}{r} = u_t(x)$$

by an easy continuity argument Consequently

$$\limsup_{\substack{r\to 0\\ r\in\mathbb{Q}}} L_r u(x) = \inf_{t\in\mathbb{Q}^+} u_t(x)$$

and from this it is clear that

$$\operatorname{Lip} u(x) \le \limsup_{\substack{r\to 0\\ r\in\mathbb{Q}}} L_r u(x).$$

On the other hand, if $\varepsilon > 0$ is arbitrary and $t \in \mathbb{Q}^+$ is such that $\limsup_{\substack{r\to 0\\ r\in\mathbb{Q}}} L_r u(x) + \varepsilon > u_t(x)$ take some $0 < t' < t$. Utilizing the monotonicity of $t \mapsto u_t(x)$ yields

$$\operatorname{Lip} u(x) \le u_{t'}(x) \le u_t(x) < \limsup_{\substack{r\to 0\\ r\in\mathbb{Q}}} L_r u(x) + \varepsilon$$

which proves the first identity. The second one is proven along the same lines. □

For Lip u there is another characterization, namely

Proposition 7.2.4. *If $u \in \mathrm{LIP}(X)$ then for every $x \in X$ Lip u is given by*

$$\operatorname{Lip} u(x) = \limsup_{r\to 0} \sup_{d(x,y)<r} \frac{|u(x)-u(y)|}{d(x,y)}.$$

Proof. That $\operatorname{Lip} u(x) \le \limsup_{r\to 0} \sup_{d(x,y)<r} \frac{|u(x)-u(y)|}{d(x,y)}$ is obvious since

$$\sup_{d(x,y)<r} \frac{|u(x)-u(y)|}{r} \le \sup_{d(x,y)<r} \frac{|u(x)-u(y)|}{d(x,y)}$$

for any $r > 0$. Let $\varepsilon > 0$ and suppose z is such that $d(x,z) < r$ and

$$\sup_{d(x,y)<r} \frac{|u(x)-u(y)|}{d(x,y)} < \frac{|u(x)-u(z)|}{d(x,z)} + \varepsilon \le \sup_{d(x,y)<r_0} \frac{|u(x)-u(y)|}{r_0} + \varepsilon$$

where $r_0 := d(x,z) < r$. The claim follows by the definition of lim sup. □

The following lemma will be used in the sequel.

Lemma 7.2.5. *For fixed $r > 0$ the map $x \mapsto \mu(B(x,r))$ satisfies*

$$\limsup_{y\to x} \mu(B(y,r)) \le \mu(B(x,r)).$$

In particular it is measurable.

Proof. Let $\delta > 0$ be arbitrary. Since for any $y \in B(x,\delta)$ one has $B(y,r) \subset B(x, r+\delta)$ it follows that

$$\sup_{y \in B(x,\delta)} \mu(B(y,r)) \leq \mu(B(x, r+\delta))$$

from which the claim follows by taking $\lim_{\delta \to 0}$. □

To connect Lip (and later lip) to the infinitesimal behaviour of Lipschitz functions the following lemma will be used. As will be seen in the last section the theorem below can be used used to prove an abstract relation which is satisfied by spaces supporting a Póincare inequality. Consequently it is ultimately a statement about the underlying metric space. The proof follows [21].

Theorem 7.2.6. *Let (X, d, μ) be a complete measure space with a doubling measure. Then there exists a constant $0 < C < \infty$ so that the inequality*

$$\frac{1}{C} \operatorname{Lip} u(x) \leq \limsup_{r \to 0} \frac{1}{r} \fint_{B(x,r)} |u - u_{B(x,r)}| \mathrm{d}\mu \leq 2 \operatorname{Lip} u(x)$$

holds for μ-almost every $x \in X$.

Proof. The righthand side inequality follows from the definition of the limit supremum: fix an arbitrary $\varepsilon > 0$. Then there exists $\delta > 0$ so that

$$\operatorname{Lip} u(x) \leq \sup_{r < \delta} \frac{1}{r} \sup_{d(y,x) < r} |u(x) - u(y)| < \varepsilon + \operatorname{Lip} u(x).$$

Thus

$$\frac{1}{r} |u(x) - u(y)| \leq \varepsilon + \operatorname{Lip} u(x)$$

for all $y \in B(x,r)$ and $r < \delta$. Now if $y, y' \in B(x,r)$ this leads to

$$\frac{1}{r} |u(y') - u(y)| \leq \frac{1}{r} |u(x) - u(y)| + \frac{1}{r} |u(x) - u(y')| \leq 2(\varepsilon + \operatorname{Lip} u(x)),$$

from which one obtains

$$\limsup_{r \to 0} \fint_{B(x,r)} |u - u_{B(x,r)}| \mathrm{d}\mu$$
$$\leq \limsup_{r \to 0} \fint_{B(x,r)} \fint_{B(x,r)} \frac{1}{r} |u(y') - u(y)| \mathrm{d}\mu(y) \mathrm{d}\mu(y') \leq 2(\varepsilon + \operatorname{Lip} u(x)).$$

This implies the rightmost inequality since $\varepsilon > 0$ is arbitrary.

To prove the first inequality it suffices to replace the $\limsup_{r \to 0}$ with $\limsup_{\substack{r \to 0 \\ r \in \mathbb{Q}}}$. This is because

$$\limsup_{\substack{r \to 0 \\ r \in \mathbb{Q}}} \frac{1}{r} \fint_{B(x,r)} |u - u_{B(x,r)}| \mathrm{d}\mu \leq \limsup_{r \to 0} \frac{1}{r} \fint_{B(x,r)} |u - u_{B(x,r)}| \mathrm{d}\mu.$$

It can also be assumed that $\mu(X) < \infty$ by dividing it into countably many balls with finite measure. Then, given $\varepsilon > 0$ Lusin's and Egoroff's theorems easily imply that there exists a set $A \subset X$ – which can be taken compact – with $\mu(X \setminus A) < \varepsilon$ and such that

(i) $\operatorname{Lip} u$ is uniformly continuous in A and

(ii) the convergence $\operatorname{Lip} u = \limsup_{\substack{r\to 0\\ r\in\mathbb{Q}}} \sup_{\substack{t<r\\ t\in\mathbb{Q}}} \sup_{d(y,\cdot)<t} \frac{|u(y)-u(\cdot)|}{d(y,\cdot)}$ is uniform in A.

(iii) the convergenge $\frac{\mu(B(x,r)\setminus A)}{\mu(B(x,r))} \overset{\substack{r\to 0\\ r\in\mathbb{Q}}}{\longrightarrow} 0$ is uniform in the set of density points of A.

Now fix $\varepsilon > 0$ and let $\delta_1 > 0$ be such that

$$\frac{|u(y)-u(z)|}{d(y,z)} < \operatorname{Lip} u(y) + \varepsilon$$

whenever $y \in A$ and $z \in B(y,\delta_1)$ (utilizing condition (ii) above). Let $\delta_2 > 0$ be such that

$$|\operatorname{Lip} u(y) - \operatorname{Lip} u(z)| < \varepsilon$$

whenever $y, z \in A$ and $d(y,z) < \delta_2$ (utilizing condition (i) above). Finally let $\delta_3 > 0$ be such that

$$\frac{\mu(B(y,r)\setminus A)}{\mu(B(y,r))} < \varepsilon$$

whenever y is a density point of A and $0 < r < \delta_3$(utilizing condition (iii) above). Now suppose $0 < r < \min\{\delta_1,\delta_2,\delta_3\} := \delta$ and fix a density point x of A. Then for any $y \in B(x,r)\cap A$ and $z \in B(y,r)\cap A$

$$|u(y)-u(z)| < d(y,z)(\operatorname{Lip} u(y)+\varepsilon) < d(y,z)(\operatorname{Lip} u(x)+2\varepsilon).$$

Integrating this yields

$$\begin{aligned}
&|u(y)-u_{B(y,r/4)}| \le \fint_{B(y,r/4)} |u(y)-u(z)|\mathrm{d}\mu(z)\\
&\le \frac{1}{\mu(B(y,r/4))}\left[\int_{B(y,r/4)\cap A} |u(y)-u(z)|\mathrm{d}\mu(z) + \int_{B(y,r/4)\setminus A} |u(y)-u(z)|\mathrm{d}\mu(z)\right]\\
&\le r/4(\operatorname{Lip} u(x)+2\varepsilon) + r/4L\frac{\mu(B(y,r/4)\setminus A)}{\mu(B(y,r/4))}\\
&\le r/4(\operatorname{Lip} u(x)+2\varepsilon) + rL\varepsilon, \qquad (7.2.1)
\end{aligned}$$

L being the Lipschitz constant of u.

Now let s denote the homogeneity exponent of μ and let $r > 0$ be such that $r(1+\varepsilon^{1/s}) < \delta$. By definition one has

$$\sup_{r'<r} \sup_{y\in B(x,r)} \frac{|u(y)-u(x)|}{d(x,y)} > \operatorname{Lip} u(x) - \varepsilon,$$

in particular there exists $r' < r, r' \in \mathbb{Q}$ and $y' \in B(x,r')$ for which

$$|u(x)-u(y')| > r'(\operatorname{Lip} u(x) - \varepsilon).$$

On the other hand the ball $B(y', 8r'\varepsilon^{1/s}) \subset B(x, r'(1+8\varepsilon^{1/s}))$ contains a point of A:

$$\frac{\mu(B(y', 8r'\varepsilon^{1/s}) \cap A)}{\mu(B(x, r'(1+8\varepsilon^{1/s})))} = \frac{\mu(B(y', 8r'\varepsilon^{1/s}))}{\mu(B(x, r'(1+8\varepsilon^{1/s})))} - \frac{\mu(B(y', 8r'\varepsilon^{1/s}) \setminus A)}{\mu(B(x, r'(1+8\varepsilon^{1/s})))}$$
$$\geq 4^{-s}\left(\frac{8\varepsilon^{1/s}}{1+8\varepsilon^{1/s}}\right)^s - \frac{\mu(B(x, r'(1+8\varepsilon^{1/s})) \setminus A)}{\mu(B(x, r'(1+8\varepsilon^{1/s})))} > \varepsilon\left(\frac{2^s}{(1+8\varepsilon^{1/s})^s} - 1\right) > 0.$$

If $y \in B(y', 8r'\varepsilon^{1/s}) \cap A$ is such a point then

$$|u(x) - u(y)| \geq |u(x) - u(y')| - |u(y) - u(y')| \geq r'(\operatorname{Lip} u(x) - \varepsilon) - 8r'L\varepsilon^{1/s}.$$

Since the set of density points of A is dense in $B(y', 8r'\varepsilon^{1/s}) \cap A$ it can be assumed that y is already a density point of A (contained in A).

These facts together with (7.2.1) yield

$$\begin{aligned}
&r'(\operatorname{Lip} u(x) - \varepsilon - 8L\varepsilon^{1/s}) < |u(y) - u(x)| \\
&\leq |u(y) - u_{B(y,r'/4)}| + |u(x) - u_{B(x,r'/4)}| + |u_{B(x,r'/4)} - u_{B(y,r'/4)}| \\
&\leq r'/2(\operatorname{Lip} u(x) + 2\varepsilon) + |u_{B(x,r'/4)} - u_{B(y,r'/4)}|,
\end{aligned}$$

or

$$(\operatorname{Lip} u(x)/2 - 3\varepsilon - 8L\varepsilon^{1/s}) \leq \frac{1}{r'}|u_{B(x,r'/4)} - u_{B(y',r'/4)}|. \tag{7.2.2}$$

The right-hand side of (7.2.2) can be estimated by

$$|u_{B(x,r'/4)} - u_{B(y',r'/4)}| \leq |u_{B(x,2r')} - u_{B(y',r'/4)}| + |u_{B(x,r'/4)} - u_{B(x,2r')}|$$

and both of the right-hand side terms of this in turn by

$$\begin{aligned}
&|u_{B(x,2r')} - u_{B(y',r'/4)}| \leq ⨍_{B(y',r'/4)} |u(z) - u_{B(x,2r')}|\,\mathrm{d}\mu(z) \\
&\leq C ⨍_{B(x,2r')} |u(z) - u_{B(x,2r')}|\,\mathrm{d}\mu(z), \\
&|u_{B(x,r'/4)} - u_{B(x,2r')}| \leq C ⨍_{B(x,2r')} |u(z) - u_{B(x,2r')}|\,\mathrm{d}\mu(z)
\end{aligned}$$

since $B(y', r'/4) \subset B(x, 2r')$. Here the constant C depends only on the doubling constant of μ.

Combining all the above results it can be concluded that for any $r < \delta$ there exists $r' < r$, $r' \in \mathbb{Q}$ so that

$$(\operatorname{Lip} u(x)/2 - 3\varepsilon - 8L\varepsilon^{1/s}) \leq \frac{C}{r'} ⨍_{B(x,2r')} |u(z) - u_{B(x,2r')}|\,\mathrm{d}\mu(z).$$

The desired inequality follows by taking $\limsup_{\substack{r\to 0\\ r\in\mathbb{Q}}}$ and letting $\varepsilon \to 0$. Since almost every point of A is a density point and A can be taken to have measure arbitrarily close to full measure the proof is complete. □

7.3 Metric measure spaces supporting a Poincaré inequality

The idea of the class of spaces about to be defined – the spaces supporting a p-Poincaré inequality – is simply to require an inequality similar to (7.1.1) to hold not only for Hajłasz upper gradients but arbitrary upper gradients. The following definition is from [14].

Definition 7.3.1. *A doubling metric measure space (X, d, μ) is said to support a (weak) p-Poincaré inequality if there are constants $C > 0$ and $\sigma \geq 1$ such that for any locally integrable Borel function $u : X \to \mathbb{R}$ and its arbitrary locally integrable upper gradient g the Poincaré inequality*

$$\fint_B |u - u_B| \mathrm{d}\mu \leq Cr \left(\fint_{\sigma B} g^p \mathrm{d}\mu \right)^{1/p} \tag{7.3.1}$$

is satisfied. In other words X supports the p-Poincaré inequality if there are constants $C > 0$ and $\sigma \geq 1$ so that for all locally integrable Borel functions the pair (u, g) satisfies the p-Poincaré inequality for any upper gradient g of u.

Note that while Theorem 7.1.1 exhibits a property of the space $M^{1,p}(X)$, definition 7.3.1 is a property concerning the underlying space X – and, in particular the space $N^{1,p}(X)$ does not appear anywhere in the definition.

This condition quantifies the idea that if a function u has some pathwise regularity (i.e. has an upper gradient controlling it along curves) then that function must have some smoothness in a more traditional sense, i.e. the mean value of it's oscillation is controlled by the same upper gradient that controls it along paths. The quantitative constant C is related to the doubling constant of the underlying measure whereas p measures how well a function can be controlled in the mean value sense in terms of the control one has over it through it's upper gradient. This connection between the behaviour of functions along curves and in the mean value sense turns out to be very fruitful for first order calculus (larger values of p correspond to less control, or a weaker connection between the two).

Example As an example it will be shown that if $C \subset \mathbb{R}^n$ is a closed convex set with positive Lebesgue measure then the metric measure space $(C, |\cdot|, \mathcal{L}^n|_C)$ supports a 1-Poincaré inequality. The measure will be abbreviated by μ. It is important to note that μ is Ahlfors n-regular, that is if $B_C(x, r)$ denotes a ball *of* C of radius $r \leq \operatorname{diam}(C)$ and centre $x \in C$ then

$$\mu(B(x, r)) \approx r^n$$

where the implied constants depend only on C and n.

Let $u : C \to \mathbb{R}$ be a locally integrable function and g it's (locally integrable) upper gradient. For any two points $y, z \in C$ the line $[y, z] = \{y + t(z - y) : t \in [0, 1]\}$ connecting them lies in C and therefore

$$|u(z) - u(y)| \leq \int_{[y,z]} g = |y - z| \int_0^1 g(y + t(z - y)) \mathrm{d}t.$$

Now let $x \in C$, $r > 0$ and let $B = B(x, r) \cap C$ be a ball of C.

The first task is to estimate $|u(y) - u_B|$ for every $y \in B$:

$$
\begin{aligned}
|u(y) - u_B| &\le \frac{1}{\mu(B)} \int_{B(y,2r)\cap C} |u(y) - u(z)|\mathrm{d}z \\
&\le \frac{2r}{\mu(B)} \int_{B(y,2r)\cap C} \int_0^1 g(y + t(z-y))\mathrm{d}t\mathrm{d}z. \qquad (7.3.2)
\end{aligned}
$$

Denote by L_t the transformation $L_t(z) = y + t(z-y)$ and note that

$$
L_t(B(y,2r)\cap C) = B(y,2tr) \cap L_t(C) \subset B(y,2tr)\cap C.
$$

Using this and the change of variables $z \mapsto L_t(z)$ yields

$$
\int_{B(y,2r)\cap C} g(y + t(z-y))\mathrm{d}t \le t^{-n} \int_{B(y,2tr)\cap C} g(z)\mathrm{d}z.
$$

Plugging this into (7.3.2) after interchanging the order of integration (justified by the local integrability of g) gives

$$
|u(y) - u_B| \le \frac{2r}{\mu(B)} \int_0^1 t^{-n} \int_{B(y,2tr)\cap C} g(z)\mathrm{d}z\mathrm{d}t.
$$

Next integrate this with respect to y (over B) to get

$$
\fint_B |u - u_B|\mathrm{d}\mu \le \frac{2r}{\mu(B)^2} \int_0^1 t^{-n} \int_{B(x,r)\cap C} \int_{B(y,2tr)\cap C} g(z)\mathrm{d}z\mathrm{d}y\mathrm{d}t. \qquad (7.3.3)
$$

To estimate this note that

$$
\begin{aligned}
&\int_{B(x,r)\cap C} \int_{B(y,2tr)\cap C} g(z)\mathrm{d}z\mathrm{d}y = \int_{B(x,r)\cap C} \int_{B(x,3r)\cap C} \chi_{B(y,2tr)\cap C}(z) g(z)\mathrm{d}z\mathrm{d}y \\
&= \int_{B(x,3r)\cap C} g(z) \int_{B(x,r)\cap C} \chi_{B(y,2tr)\cap C}(z)\mathrm{d}y\mathrm{d}z \\
&= \int_{B(x,3r)\cap C} g(z)\mathcal{L}^n(B(z,2tr)\cap B(x,r)\cap C)\mathrm{d}z \lesssim (tr)^n \int_{B(x,3r)\cap C} g(z)\mathrm{d}z.
\end{aligned}
$$

This combined with (7.3.3) then gives the estimate

$$
\fint_B |u - u_B|\mathrm{d}\mu \lesssim \frac{2r}{\mu(B)^2} \int_0^1 r^n \int_{B(x,3r)\cap C} g(z)\mathrm{d}z\mathrm{d}t \approx r \fint_{3B} g\mathrm{d}\mu.
$$

Here $3B = B(x,3r)\cap C$.

The above reasoning is actually valid for any Ahlfors n-regular measure on C. It does not, however, extend directly to non-convex domains, and in general the claim fails for arbitrary domains D in $\mathbb{R}^n$ when the measure is the restriction of $\mathcal{L}^n$ to D. [17]

Continuing with general spaces X it is not difficult to see that if $u \in \mathrm{LIP}(X)$ then $\mathrm{lip}(u)$ is an upper gradient of u.

Proposition 7.3.2. *Let (X,d,μ) be a metric measure space and $u : X \to \mathbb{R}$ a Lipschitz function. Then $\operatorname{lip} u$ is an upper gradient of u.*

Proof. Let $\gamma : [0, L] \to X$ be the arc-length parametrization of a rectifiable curve (if no rectifiable curves exist then any function is an upper gradient of u) and consider the function $u \circ \gamma : [0, L] \to \mathbb{R}$. This obeys

$$u(\gamma(L)) - u(\gamma(0)) = \int_0^L \frac{\mathrm{d}}{\mathrm{d}t} u \circ \gamma(s) \mathrm{d}s,$$

in particular

$$|u(\gamma(L)) - u(\gamma(0))| \leq \int_0^L \left| \frac{\mathrm{d}}{\mathrm{d}t} u \circ \gamma(s) \right| \mathrm{d}s.$$

But at any point $s \in [0, L]$ of differentiability

$$\begin{aligned} \left| \frac{\mathrm{d}}{\mathrm{d}t} u \circ \gamma(s) \right| &= \liminf_{h \to 0} \frac{|u(\gamma(s+h)) - u(\gamma(s))|}{|h|} \\ \leq \liminf_{h \to 0^+} \sup_{y \in B(\gamma(s), h)} &\frac{|u(y) - u(\gamma(s))|}{h} = \operatorname{lip} u(\gamma(s)). \end{aligned}$$

This is since $d(u(\gamma(s+h)), u(\gamma(s))) \leq |h|$ so always $u(\gamma(s+h)) \in B(u(\gamma(s)), |h|)$. Consequently

$$|u(\gamma(L)) - u(\gamma(0))| \leq \int_\gamma \operatorname{lip} u.$$

□

A natural question is whether or not $\operatorname{lip} u$ (or Lip) is in some sense an optimal upper gradient. Both indeed turn out to be "optimal" when the underlying space supports a Poincaré inequality, in the sense that there is a constant so that any upper gradient times the constant is a pointwise upper bound of Lip up to a set of measure zero. These special upper gradients are also adequate for the definition of a space supporting a Poincaré inequality: in [21] a definition using lip is presented and these two definitions coincide at least whenever the measure μ is doubling and the space complete.

Theorem 7.3.3. *A complete metric measure space X, whose measure μ is doubling, supports the p-Poincaré inequality if and only if there are $C > 0$ and $\sigma \geq 1$ so that*

$$⨍_B |u - u_B| \mathrm{d}\mu \leq Cr \left(⨍_{\sigma B} (\operatorname{lip} u)^p \mathrm{d}\mu \right)^{1/p}$$

for any Lipschitz function u and any ball B of radius r.

The proof of this theorem can be found in for instance in [20].

Definition 7.3.4. *A metric space X is said to be C-quasiconvex, $C > 0$ a constant, if it is path-connected and for any two points $x, y \in X$ there exists a rectifiable curve γ joining them so that*

$$\ell(\gamma) \leq C d(x, y).$$

When the particular constant is unimportant it is said merely that X is quasiconvex.

Theorem 7.3.5. *A complete path connected space X supporting a Póincare inequality for all pairs (u,g) of a Lipschitz function and a* continuous *upper gradient g of u is quasiconvex.*

Proof. Let $\gamma:[0,1]\to X$ be a path (not necessarily rectifiable!) connecting any two points x and y. For each $k\in\mathbb{N}$ and any partition $\tau=\{0=t_0<\dots t_n=1\}$ of $[0,1]$ set

$$s_k^\tau(\gamma)=\sum_{i=1}^{n}\min\{\ell(\gamma|_{[t_{i-1},t_i]}),kd(\gamma(t_i),\gamma(t_{i-1}))\}.$$

Further let $\ell_k(\gamma)=\inf_\tau s_k^\tau(\gamma)$. Notice that if γ happens to be rectifiable then $s_k^\tau\le\sum_{i=1}^{n}\ell(\gamma|_{[t_{i-1},t_i]})=\ell(\gamma)$ so that $\ell_k\le\ell$. Also $\ell_k(\gamma)\le kd(\gamma(0),\gamma(1))=kd(x,y)$. Finally when $k\ge1$, by $\min\{\ell(\gamma|_{[t_{i-1},t_i]}),kd(\gamma(t_i),\gamma(t_{i-1}))\}\ge d(\gamma(t_i),\gamma(t_{i-1}))$ and the triangle inequality it follows that

$$s_k^\tau\ge\sum_{i=1}^{n}d(\gamma(t_i),\gamma(t_{i-1}))\ge d(x,y).$$

Fix an arbitrary point $x_0\in X$. Define $u_k(x)=\inf_\gamma\ell_k(\gamma)$ where the infimum is taken over all paths joining x_0 and x, for arbitrary $x\in X$. The observations made above yield $d(x_0,x)\le u_k(x)\le kd(x_0,x)$ for $k\ge1$. Let $\varepsilon>0$ and $x,y\in X$ and assume, as can be done without loss of generality, that $u_k(x)>u_k(y)$. Let γ_y be a path joining x_0 and y such that $\ell_k(\gamma_y)-\epsilon<u_k(y)$. Let γ_x be any path joining x_0 and x and γ a path from x to y. Then

$$u_k(x)-u_k(y)\le\ell_k(\gamma_x)-\ell_k(\gamma_y)+\varepsilon\le\ell_k(\gamma)+\varepsilon.$$

Since ε is arbitrary and $\ell_k(\gamma)\le kd(x,y)$ the k-Lipschitz continuity of u_k follows. From the same inequality it follows that $g\equiv1$ is an upper gradient for u_k, for any k. (Since $\int_\gamma 1=\ell(\gamma)\ge\ell_k(\gamma)$.) Employing corollary 7.1.2 it then follows that

$$|u_k(x)-u_k(y)|\le C[\mathcal{M}g(x)+\mathcal{M}g(y)]d(x,y)=2Cd(x,y)$$

almost everywhere and hence everywhere. Thus there is a uniform Lipschitz constant for u_k with respect to k. If, in particular y is chosen to be x_0 the inequality

$$u_k(x)\le Cd(x_0,x) \tag{7.3.4}$$

is obtained. Hence there also exists a constant $C>0$ (perhaps different from that in equation (7.3.4)) – depending only on the constants in the Poincaré inequality – so that for every k there is a path γ_k for which

$$\ell_k(\gamma_k)\le Cd(x_0,x)$$

and, consequently a constant $C>0$ so that for every k there corresponds a partition τ_k for which

$$s_k:=s_k^{\tau_k}(\gamma_k)\le Cd(x_0,x).$$

Consider the segments $[t_{k,i-1},t_{k,i}]$, $i=1,\dots,n(k)$ determined by the partitions τ_k. Construct a new metric space as follows. For each index-pair k,i, if

$$\ell(\gamma_k|_{[t_{k,i-1},t_{k,i}]})>kd(\gamma_k(t_{k,i-1}),\gamma_k(t_{k,i}))=:kd(x_{k,i-1},x_{k,i}) \tag{7.3.5}$$

then add a "line segment" $I_{k,i}$ connecting the points $x_{k,i-1}$ and $x_{k,i}$, equipped with the metric l inherited from the real line multiplied by a factor of $\frac{d(x_{k,i-1}, x_{k,i}))}{t_{k,i} - t_{k,i-1}}$. Thus the line segment has length $d(x_{k,i-1}, x_{k,i})$. Denote by $\widetilde{X}$ the result of this process after going through every k and i. For two arbitrary points $x, y \in \widetilde{X}$ define a distance by

$$\tilde{d}(x,y) = \begin{cases} d(x,y) & \text{if } x,y \in X \\ l(x,y) & \text{if } x \text{ and } y \text{ are in the same line-segment} \\ \min\{l(x,x_{k,i}) + d(x_{k,i},y), \\ \quad l(x,x_{k,i-1}) + d(x_{k,i-1},y)\} & \text{if } x \in I_{k,i} \text{ and } y \in X \\ \min\{l(x,x^*) + d(x^*,y^*) + l(y^*,y) : \\ x^* = x_{k,i}, x_{k,i-1}; y^* = x_{r,j}, x_{r,j-1}\} & \text{if } x \in I_{k,i}, y \in I_{r,j}, \\ & (k,i) \neq (r,j). \end{cases}$$

That this indeed defines a metric is an easy albeit somewhat lengthy calculation. (Of course, the case where $y \in I_{k,i}$ and $x \in X$ is defined likewise as in above.) In $\widetilde{X}$ replace the curves γ_k by $\tilde{\gamma}_k$ where

$$\tilde{\gamma}_k|_{[t_{k,i-1},t_{k,i}]} = \gamma_k|_{[t_{k,i-1},t_{k,i}]}$$

if (7.3.5) doesn't hold. Otherwise $\tilde{\gamma}_k|_{[t_{k,i-1},t_{k,i}]}$ consists of the line segment $I_{k,i}$ from $x_{k,i-1}$ to $x_{k,i}$. Therefore

$$\tilde{\ell}(\tilde{\gamma}_k) \leq s_k \leq Cd(x_0, x)$$

and consequently $kl(I_{k,i}) \leq \tilde{\ell}(\tilde{\gamma}_k) \leq Cd(x_0,x)$. Note that $\tilde{\ell}$ is used to denote the lenght in the new metric $\tilde{d}$ and l the usual euclidean length/metric.

The new space $\widetilde{X}$ is proper. To see this it is convenient to demonstrate first that it is complete. To this end let $(x_m) \subset \widetilde{X}$ be a Cauchy sequence. If infinitely many of its element lie in either X or in finitely many of the segments $I_{k,i}$ (whence infinitely many elements lie in *one* segment $I_{k,i}$) then it is clear that the sequence is convergent (because X and the line segments are complete). Hence it may be assumed that there is a subsequence, relabeled (x_m) so that $x_m \in I_{k_m,i_m}$ where the line-segments are pairwise disjoint. But then

$$\operatorname{dist}(I_{k_m,i_m}, I_{k_l,i_l}) \leq \tilde{d}(x_m, x_l).$$

Here

$$\operatorname{dist}(I_{r,j}, I_{k,i}) := \min\{d(x_{r,j}, x_{k,i}), d(x_{r,j-1}, x_{k,i}), d(x_{r,j}, x_{k,i-1}), d(x_{r,j-1}, x_{k,i-1})\}.$$

This implies that, passing to a subsequence if necessary, either x_{k_m,i_m} or x_{k_m,i_m-1} is a Cauchy sequence whence there is a limit $x' \in X$.

$$\limsup_{m\to 0} \tilde{d}(x_m, x') = \limsup_{m\to 0} l(x_m, x_{k_m,*}) \leq \limsup_{m\to\infty} \tilde{d}(x_m, x_l)$$

for every l. This proves $x_m \to x'$ in $\tilde{d}$ and hence the completeness of $\widetilde{X}$.

Now assume that $\widetilde{X}$ is not proper – let $B = \widetilde{B}(x,r)$ be a closed ball of $\widetilde{X}$ which is not compact. Then there is a sequence $(x_n) \subset B$ and $a > 0$ so that

$\tilde{d}(x_n, x_m) \geq a$ if $n \neq m$. Since X is proper all but a finite amount of the points in the sequence must lie in the line segments and further no line segment can contain more than a finite number of these points. Hence there is a subsequence (x_n) so that $x_n \in I_{k_n,i_n}$ with $k_1 < k_2 < \ldots$. From the expression of $\tilde{d}$ it can be estimated that

$$0 < a \leq \tilde{d}(x_n, x_m) \leq l(I_{k_n,i_n}) + l(I_{k_m,i_m}) + d(x_{k_n,i_n}, x_{k_m,i_m}).$$

But since $x_{k_n,i_n} \in X$ there is a convergent subsequence. Hence the first two terms in the inequality above must stay above a fixed positive number. This, however, is not possible either since the number of the segments is infinite and $l(I_{k,i}) \leq C/k$. This finishes the argument that $\widetilde{X}$ is proper.

As a result of this trick the situation is now such that there is a sequence of curves $\tilde{\gamma}_k$ parametrized from 0 to 1 and obeying $\tilde{\ell}(\tilde{\gamma}_k) \leq Cd(x_0, x)$. The next task is to substract a subsequence that converges, in a certain sense, to a curve $\tilde{\gamma}$ having the property $\tilde{\ell}(\tilde{\gamma}) \leq Cd(x_0, x)$. Begin by observing that if the curves are parametrized by arc-length and then these parametrizations are re-scaled back to $[0, 1]$ then all the parametrizations are $\tilde{\ell}(\tilde{\gamma}_k)$-Lipschitz where $\tilde{\ell}(\tilde{\gamma}_k) \leq Cd(x_0, x)$ – the uniform Lipschitz constant implying, in particular, the equicontinuity of the sequence. Next let $A = \{q_1, q_2, \ldots\}$ be a countable dense subset of $[0, 1]$ and denote $a_{k,n} = \tilde{\gamma}_k(q_n)$. For $(a_{k,1})$ choose a subsequence converging to $a_1 \in \widetilde{X}$. From this subsequence take another, so that $a_{k,2} \to a_2$ and so on. Define the function $\tilde{\gamma}$ on the dense subset A by $\tilde{\gamma}(q_n) := a_n$. If $q_n, q_m \in A$ with $n > m$ then using the uniform Lipschitz constant we get

$$d(\tilde{\gamma}(q_n), \tilde{\gamma}(q_m)) = \lim_{k\to\infty} d(\tilde{\gamma}_k(q_n), \tilde{\gamma}_k(q_m)) \leq C|q_n - q_m|.$$

The limit is with respect to the subsequence for the nth step in the above construction. This estimate and the completeness of $\widetilde{X}$ allow for an arbitrary $t \in [0, 1]$ the following definition: $\tilde{\gamma}(t) := \lim_{n\to\infty} \tilde{\gamma}(q_n)$ where (q_n) is any sequence converging to t. This is of course well defined. The resulting function $\tilde{\gamma}$ is a $Cd(x_0, x)$-Lipschitz curve. Thus for any partition $\tau = \{0 = t_0 < \cdots < t_n = 1\}$

$$\sum_{i=1}^{n} \tilde{d}(\tilde{\gamma}(t_i), \tilde{\gamma}(t_{i-1})) \leq Cd(x_0, x) \sum_{i=1}^{n} (t_i - t_{i-1}) = Cd(x_0, x)$$

and hence $\tilde{\ell}(\tilde{\gamma}) \leq Cd(x_0, x)$.

It remains to show that $\tilde{\gamma}$ actually lies in X (from which it follows that $\tilde{\ell}(\tilde{\gamma}) = \ell(\tilde{\gamma})$). Suppose $\tilde{\gamma}(t) \in I_{k,i}$ for some $t \in [0, 1], k, i$. By the continuity of $\tilde{\gamma}$ it must attain all the values in the segment. However each of the curves $\tilde{\gamma}_r$ attains values only in segments $I_{r,i}$ whose length converges to zero as $r \to \infty$. Now if $\tilde{\gamma}$ attains values in some $I_{k,i}$ then for sufficiently large the curves γ_r must also attain values in $I_{k,i}$ which was observed to be impossible. Thus $\tilde{\gamma}$ cannot attain values in any segment of fixed positive length and hence $\tilde{\gamma}(t) \in X$ for every t. The quasiconvexity of X follows. □

The proof of this result is taken from [13]. Here the path connectedness of X was part of the assumptions. In [14] the quasiconvexity is proven without this assumption. In fact spaces supporting a Poincaré inequality are much more than quasiconvex. In such spaces there are always ample curves joining two given points, excluding in particular any behaviour akin to the Cantor set in section three.

7.4 Coincidence of $N^{1,p}(X)$ and $M^{1,p}(X)$

Throughout this subsection it will be assumed that, in addition to being complete and doubling, the metric measure spaces (X, d, μ) appearing below admit a p-Poincaré inequality

$$\fint_B |u - u_B| \mathrm{d}\mu \leq C_P r \left(\fint_{\sigma B} g^p \mathrm{d}\mu \right)^{1/p} \tag{7.4.1}$$

for some $p > 1$. The aim of this subsection is to prove the isomorphy of the spaces $N^{1,p}(X)$ and $M^{1,p}(X)$. Note, however that the case $p = 1$ is not included in the discussion as, indeed, even in the model case of $X = \mathbb{R}^n$ the coicidence of these two spaces is not true. (In this case $N^{1,1}$ coincides with the classical Sobolev space and $M^{1,1}$ is a smaller space, namely the Hardy Sobolev space over $\mathbb{R}^n$. For details see [23].)

It is easy to demonstrate a partial result akin to the aim of this subsection. The essential work has been done in the two previous sections and what remains is to connect the dots. Recall Theorem 6.4.5 which states that $M^{1,p}(X)$ embeds continuously into $N^{1,p}(X)$. If p is as in (7.3.1) then X satisfies the q-Poincaré inequality with the same constants for any $q \geq p$ (by Jensen's inequality). Thus by Theorem 6.4.5 one can deduce that $M^{1,q}(X) \hookrightarrow N^{1,q}(X)$ continuously for all such q. Next, for any $q > p$ consider any function $u \in L^q(X)$ and its upper gradient $g \in L^q(X)$ (so that in fact $u \in N^{1,q}(X)$). Recall, from Corollary 7.1.2 that, since the pair (u, g) now satisfies the p-Poincaré inequality

$$\fint_B |u - u_B| \mathrm{d}\mu \leq Cr \left(\fint_{\sigma B} g^p \mathrm{d}\mu \right)^{1/p}$$

for every ball, then almost everywhere the pointwise estimate

$$|u(x) - u(y)| \leq Cd(x,y)[\left(\mathcal{M}_{2\sigma d(x,y)} g^p(x)\right)^{1/p} + \left(\mathcal{M}_{2\sigma d(x,y)} g^p(y)\right)^{1/p}]$$

holds so $(\mathcal{M}g^p)^{1/p}$ is a Hajłasz gradient for u. Since $q > p > 1$ it follows that

$$||(\mathcal{M}g^p)^{1/p}||_q \leq C||g^p||_{q/p} \leq C||g||_q$$

and hence

$$||u||_{M^{1,p}(X)} \leq ||u||_{L^p(\mu)} + \inf_{g \in G(u)} ||\mathcal{M}g||_p \leq ||u||_p + C \inf_{g \in G(u)} ||g||_p \leq C||u||_{N^{1,p}(X)},$$

that is, $N^{1,q}(X)$ embeds continuously into $M^{1,q}(X)$. Note that this result only holds for p strictly greater than 1 as the maximal operator is not bounded from $L^1 \to L^1$.

Combining the above reasoning and 6.4.5 yields

Theorem 7.4.1. *With fixed $p > 1$ in (7.4.1) the Hajłasz space $M^{1,q}(X)$ is isomorphic to $N^{1,q}(X)$ for every $p < q < \infty$.*

With the aid of the following notable result the end-point case of the above theorem can be addressed.

Theorem 7.4.2. *If the complete doubling metric space (X, d, μ) admits a p-Poincaré inequality for some $p > 1$ then there exists some $0 < \varepsilon < p - 1$ so that (X, d, μ) admits a $p - \varepsilon$-Poincaré inequality (and consequently a q-Poincaré inequality for any $q > p - \varepsilon$).*

The proof of this will be omitted; it can, however be found from [22]. Now it easily follows that

Corollary 7.4.3. *with fixed $p > 1$ in (7.4.1) the Hajłasz space $M^{1,p}(X)$ is isomorphic to $N^{1,p}(X)$.*

Proof. Apply 7.4.1 to $p - \varepsilon$. □

8 A differentiable structure for spaces supporting a Poincaré inequality

In the beginning of the previous section it was stated that in spaces supporting a Poincaré inequality a notion of differentiability akin to the Euclidean case arises. It is the purpose of this section to define what exactly that notion is and then to prove that, indeed, in these spaces such a notion is possible and is possessed of many nice properties.

The Poincaré inequality was already linked to the infinitesimal behaviour of Lipschitz functions – in particular to the operator lip – through theorem 7.3.3. For the rest of this section the Poincaré property will be replaced by another one using Lip and lip. Throughout this section (X, d, μ) will *always* be assumed to be complete and doubling. The rest of this exposition is based on the paper of Stephen Keith, [21].

Theorem 8.0.4. *Suppose (X, d, μ) supports a p-Poincaré inequality for some $p \geq 1$. Then there exists a constant $K > 0$ depending only on the constant C_P in the Poincaré inequality so that for each $u \in \mathrm{LIP}(X)$*

$$\operatorname{Lip} u(x) \leq K \operatorname{lip} u(x) \text{ for almost every } x \in X.$$

Proof. Fix $u \in \mathrm{LIP}(X)$. According to Proposition 7.2.6 there is a constant $C > 0$ such that

$$\frac{1}{C} \operatorname{Lip} u(x) \leq \limsup_{r \to 0} \frac{1}{r} \fint_{B(x,r)} |u - u_{B(x,r)}| \mathrm{d}\mu$$

for almost every $x \in X$. Since $\operatorname{lip} u$ is an upper gradient of u (Proposition 7.3.2 it follows that

$$\frac{1}{r} \fint_{B(x,r)} |u - u_{B(x,r)}| \mathrm{d}\mu \leq C_P \left(\fint_{B(x,\sigma r)} (\operatorname{lip} u)^p \mathrm{d}\mu \right)^{1/p}$$

for every $x \in X$ and $r > 0$. Lebesgue's differentiation theorem then asserts that for almost every $x \in X$

$$\begin{aligned}\frac{1}{C}\operatorname{Lip} u(x) &\leq \limsup_{r\to 0}\frac{1}{r}\fint_{B(x,r)}|u-u_{B(x,r)}|\mathrm{d}\mu\\ &\leq C_P\limsup_{r\to 0}\left(\fint_{B(x,\sigma r)}(\operatorname{lip} u)^p\mathrm{d}\mu\right)^{1/p} = C_P\operatorname{lip} u(x).\end{aligned}$$

Hence $K = C_P C$. □

This more elegant condition leads to a natural notion of generalized linearity of tangent functions on tangent spaces at a given point $x \in X$ through which the differential structure (to be shortly defined) will be developed.

Definition 8.0.5. *Given a metric measure space (X, d, μ) and a vector space $V \subset LIP(X)$, a denumerable collection (X_k, φ_k), $k \in K$ of pairs consisting of a measurable set X_k and a function*

$$\varphi_k = (\varphi_k^1, \ldots, \varphi_k^{n(k)}) : X_k \to \mathbb{R}^{n(k)}$$

where $\varphi_k^i \in V$, $1 \leq i \leq n(k)$ is said to be a (strong) measurable structure with respect to V if

1. *each X_k has positive measure and $X = \bigcup_{k\in K} X_k$.*
2. *There exists a natural number n (possibly zero) such that $0 \leq n(k) \leq n$ for every $k \in K$.*
3. *For any function $u \in V$ and every $k \in K$ there exists a measurable function $\mathrm{d}_k u : X_k \to \mathbb{R}^{n(k)}$ for which*

$$\lim_{\substack{y\to x\\ y\neq x}}\frac{|u(y)-u(x)-\mathrm{d}_k u(x)\cdot(\varphi_k(x)-\varphi_k(y))|}{d(y,x)} = 0 \tag{8.0.2}$$

   for μ-almost every $x \in X_k$.

Furthermore, the sets X_k can be taken so that the following holds: there exist positive numbers $\delta_k > 0$ so that for each $k \in K$ and almost every $x \in X_k$ the inequality

$$\operatorname{Lip}(\lambda\cdot\varphi_k)(x) \geq \delta_k|\lambda|$$

holds for all $\lambda \in \mathbb{R}^{n(k)}$. Here it is understood that $|\lambda|$ refers to the standard Euclidean norm of the vector λ.

As one might guess, not every metric measure space X and every $V \subset \operatorname{LIP}(X)$ can be equipped with a differential structure of this sort. One such example, when $V = \operatorname{LIP}(X)$, is given by the Cantor set appearing in section 3. [3] The purpose of this subsection and a main purpose of the whole exposition is to demonstrate that metric measure spaces supporting a p-Poincaré inequality

[3] If the Cantor set C *could* be equipped with a differential structure as defined above then according to the main result of this section the Hajłasz space over C would be reflexive. This, however was seen not to be the case.

do admit such a differential structure with respect to the vector space of all Lipschitz functions.

Before introducing some machinery used in sequel a few remarks on Definition 8.0.5 are in order.

i) The differential structure is called degenerate if $n(k) = 0$ for some $k \in K$. If it is not degenerate then it is called non-degenerate.

ii) If x is an isolated point then equation (8.0.2) does not pose any restrictions.

iii) The smallest integer n satisfying $n(k) \leq n$ for all $k \in K$ is referred to as the dimension of the differentiable structure.

iv) The last condition is a technical one, added for later convenience. It does not appear in the definition in [21].

8.1 Tangent spaces

One way to get to the concepts in definition 8.0.5, pursued in this paper, is to analyze the existence and properties of tangent spaces (and functions) at a given point. This seems natural enough given the similarity of these concepts to the classical manifold structure. However, the tangent spaces can no longer be given such a direct definition as in the classical case and therefore will have to be aproached through the more general notion of convergence of metric spaces. This subsection then starts with a (rather long and boring) list of definitions followed by a few results that will be helpful in the context at hand.

Definition 8.1.1. *Let (Z, d) be a complete locally compact metric space and F_n, F non-empty closed subsets of Z. F_n is said to converge to F, abbreviated $F_n \to F$, if*

$$\lim_{n\to\infty} \sup_{z\in F_n \cap B(q,R)} d(z, F) = 0 \text{ and}$$
$$\lim_{n\to\infty} \sup_{z\in F \cap B(q,R)} d(z, F_n) = 0$$

for every $q \in Z$ and $R > 0$.

(If it so happens that some of the sets $F_n \cap B(q, R), F \cap B(q, R)$ vanish then the suprema are interpreted to equal zero.) Convergence by this definition means a sort of "local Hausdorff" convergence of the sets in question.

Definition 8.1.2. *Let (X, d) and (Y, g) be complete locally compact metric spaces, F_n and F nonempty closed subsets of X, G_n and G nonempty closed subsets of Y and $f_n : F_n \to G_n$, $f : F \to G$ functions. The sequence f_n is said to converge to f if $F_n \to F$ and $G_n \to G$ in the sense of definition 8.1.1 and the following holds: whenever $x_n \in F_n$ is a sequence for which there exists $x \in F$ so that $x_n \to x$ the values of x_n under f_n converge to $f(x)$, that is $g(f_n(x_n), f(x)) \to 0$ as $n \to \infty$.*

The next definition concerns two properties of a sequence of (not necessarily convergent) functions. Sequences satifying these are later seen to have nice properties.

Definition 8.1.3. *Let $f_n : F_n \to G_n$ be a sequence of functions, $F_n \subset (X, d)$, $G_n \subset (Y, g)$ nonempty closed subsets of the complete locally compact metric spaces (X, d) and (Y, g). The sequence (f_n) is said to be* equicontinuous *if for every $\varepsilon > 0$ there exists some $\delta > 0$ so that whenever $x, y \in F_n$, $d(x, y) < \delta$ then $g(f_n(x), f_n(y)) < \varepsilon$.*

Furthermore (f_n) is said to be uniformly bounded on bounded sets *if*

$$\sup_n \sup_{z,w \in F_n \cap B(x_n,R)} g(f_n(z), f_n(w)) < \infty$$

for all $x_n \in F_n$ and $R > 0$.

For the next definition it is useful to recall that if (X, d) is any metric space and $0 < \alpha \leq 1$, then the space (X, d^α), where $d^\alpha(x, y) = d(x, y)^\alpha$, is again a metric space, called the snowflaked (or α-snowflaked) version of X.

Definition 8.1.4. *A sequence (X_n, d_n) of complete locally compact metric spaces is said to converge to a complete locally compact metric space (X, d) if there exists some $m \in \mathbb{N}$ and some $\alpha \in (0, 1]$ so that there are C-bi-Lipschitz embeddings $\imath_n : (X_n, d_n^\alpha) \to (\mathbb{R}^m, |\cdot|)$, $\imath : (X, d^\alpha) \to (\mathbb{R}^m, |\cdot|)$ for some $C > 0$ so that $\imath_n(X_n) \to \imath(X)$ as subsets of $\mathbb{R}^m$. In addition the following property should hold. If $x_n, y_n \in X_n$ and $x, y \in X$ are such that $|\imath_n(x_n) - \imath(x)| \to 0$ and similarly for y then $d_n(x_n, y_n) \to d(x, y)$. In other words $d_n \to d$ in the sense of definition 8.1.2 (with respect to the metric space $(\imath(X) \times \imath(X), d = \sqrt{d_1^2 + d_2^2})$).*

The motivation behind this definition is, in part, Assouad's embedding theorem. This remarkable and beautiful result and its proof can be found for instance in [17].

Theorem 8.1.5. *(Assouad's embedding theorem.) Let (X, d) be a complete and doubling space. Then for every $0 < \alpha < 1$ there corresponds a natural number $m(\alpha)$ so that there is a bi-Lipschitz mapping from (X, d^α) to $\mathbb{R}^{m(\alpha)}$ (which is equipped with the usual Euclidean metric), that is, (X, d^α) is bi-Lipschitz embeddable into $(\mathbb{R}^{m(\alpha)}, |\cdot|)$.*

A pointed metric space is a triplet (X, d, x) where (X, d) is a metric space and x some designated point of X.

Definition 8.1.6. *A sequence (X_n, d_n, p_n) of complete locally compact pointed metric spaces is said to converge to a complete locally compact pointed metric space (X, d, p) if $(X_n, d_n) \to (X, d)$ in the sense of 8.1.4 and the embeddings in question can be chosen so as to respect the basepoints, that is $\imath_n(p_n) = q = \imath(p)$.*

Definition 8.1.7. *Suppose (X, d, x) is a complete locally compact pointed metric space and f a real valued function on X. The triplet (Z, ρ, z) equipped with a function $g : Z \to \mathbb{R}$ is a tangent space for (X, d, x) and g a tangent function of f if there exists a sequence $r_n > 0$ of real numbers converging to zero and a compact set K containing a neighbourhood of x so that $(K, d/r_n, x)$ converges to (Z, ρ, z) in the sense of definition 8.1.6 and if, in addition,*

$$f_n \circ \imath_n^{-1} \to g \circ \imath^{-1}$$

in the sense of definition 8.1.2. Here $f_n(\cdot) = \dfrac{f(\cdot) - f(x)}{r_n}$ and $\imath_n$ and $\imath$ are the respective embeddings for $(K, (d/r_n)^\alpha, x)$ and (Z, ρ^α, z) provided by definition 8.1.6.

Instead of talking about pointed metric spaces (X, d, x) and a function $f : X \to \mathbb{R}$ separately it is more convenient to talk about the quartet (X, d, x, f) which will from now on be referred to as a space-function in accordance with [21]. Also the tangent space (Z, ρ, z) is said to be *subordinate* to (r_n). The class of all tangent functions for a given space-function (X, d, x, f) is denoted by $T(X, d, x, f)$.

Lemma 8.1.8. *Let (X_n, d_n) be a sequence of complete locally compact metric spaces with limit (X, d). Let $(\mathbb{R}^m, |\cdot|)$ be the mutual space, $\alpha \in (0, 1]$ the constant appearing in definition 8.1.6 and $\imath_n, \imath$ the respective embeddings of (X_n, d_n^α) and (X, d^α) into $\mathbb{R}^m$. If (x_n) is a sequence such that $x_n \in X_n$ and $z \in \mathbb{R}^m$ satisfies $|\imath_n x_n - z| \overset{n\to\infty}{\to} 0$ then there is some $x \in X$ so that $z = \imath(x)$.*

Proof. From the definition of convergence and the assumptions above one has

$$\lim_{n\to\infty} \operatorname{dist}(\imath_n x_n, \imath X) \leq \lim_{n\to\infty} \sup_{y \in \imath_n X_n \cap B(z,R)} \operatorname{dist}(y, \imath X) = 0$$

for any $R > 0$. Now since $\operatorname{dist}(\cdot, \imath X)$ is a continuous mapping it follows that $\operatorname{dist}(z, \imath X) = 0$ which, by the closedness of $\imath X$ implies $z \in \imath X$. $\square$

Lemma 8.1.9. *Let F_n be a sequence of non-empty closed subsets of $\mathbb{R}^m$ and suppose that there exists $r > 0$ so that*

$$F_n \cap B(0, r) \neq \varnothing$$

for all sufficiently large n. Then there is a convergent subsequence of F_n.

For a proof see [4].

Proposition 8.1.10. *Let F_n, F, a and F' be nonempty closed subsets of a proper space (Z, d) such that $F_n \to F$ and $F_n \to F'$. Then $F = F'$.*

To prove this the following lemma will be used.

Lemma 8.1.11. *Let F_n be a convergent sequence of nonempty closed subsets of a proper space (Z, d). Then for every x in the limit of the sequence and every $R > 0$ there is n_0 so that $F_n \cap B(x, R) \neq \varnothing$ for every $n \geq n_0$.*[4]

Proof. Let $F = \lim_{n\to\infty} F_n$ in the sense of definition 8.1.1 and let $x \in F$, $R > 0$ be arbitrary. Since

$$\lim_{n\to\infty} \sup_{z \in F \cap B(x,R/2)} d(z, F_n) = 0$$

and $F \cap B(x, R/2) \neq \varnothing$ there are sequences $(z_n) \subset F \cap B(x, R/2)$ and $y_n \subset F_n$ such that $d(z_n, y_n) \to 0$. On the other hand by the compactness of $F \cap B(x, R/2)$ there is a subsequence (z_n) and $z_R \in F \cap B(x, R/2)$ so that $z_n \to z_R$. But then also $y_n \to z_R$, in particular there exists n_0 so that $d(x, y_n) \leq d(x, z_R) + d(z_R, y_n) \leq R/2 + R/2$ whenever $n \geq n_0$. Hence every $n \geq n_0$ $y_n \in F_n \cap B(x, R)$. $\square$

[4] In particular this implies the existence of a sequence $x_n \in F_n$ such that $x_n \to x$ in Z. This will be used repeatedly in the sequel.

Proof of 8.1.10. Take any $x \in F$ and $R > 0$. Since

$$\lim_{n\to\infty} \sup_{z\in F_n\cap B(x,R)} d(z, F') = 0$$

and $F_n \cap B(x, R) \neq \varnothing$ for all but a finitely many n there exist sequences (z_n) and (y_n) with $z_n \in F_n \cap B(x, R)$, $y_n \in F'$ such that

$$d(y_n, z_n) \leq 2 \sup_{z\in F_n\cap B(x,R)} d(z, F') \to 0.$$

Again by the compactness-argument there is $x_R \in B(x, R)$ and subsequences so that $z_n \to x_R$ and, consequently, $y_n \to x_R$. This implies $x_R \in F'$ since F' is closed. By taking $R = 1/k$ for each $k \in \mathbb{N}$ a sequence $(x_k) \subset F'$ is obtained, and since $x_k \in B(x, 1/k)$ the sequence converges to x which then belongs to F'. Thus $F \subset F'$ and by reversing the roles of F and F' in the argument the opposite inclusion is obtained. □

Proposition 8.1.12. *(Equivalence of limits of metric spaces.) Let (X, d) and (Y, g) both be limits of a sequence of proper metric spaces (X_n, d_n). Then there is a bijective isometric map $\phi : X \to Y$.*

Proof. Let $\imath'_n : (X_n, (d_n)^\alpha) \to \mathbb{R}^m$, $\imath' : (X_n, (d_n)^\alpha) \to \mathbb{R}^m$ be the C-Lipschitz embeddings for some α and m provided by the definition of the convergence of metric spaces, so that $\imath'_n X_n \to \imath' X$. Let $\jmath'_n : (X_n, (d_n)^\beta) \to \mathbb{R}^k$ and $\jmath' : (Y, g^\beta) \to \mathbb{R}^k$ be similarly for possibly different k and β. Let τ_1 be the bi-Lipschitz embedding of $(\mathbb{R}^m, |\cdot|^\beta)$ into some $(\mathbb{R}^l, |\cdot|)$ provided by Assouad's embedding Theorem and likewise, τ_2 the bi-Lipschitz embedding of $(\mathbb{R}^k, |\cdot|^\alpha)$ into some $(\mathbb{R}^s, |\cdot|)$. It can be assumed that $l = s$ by replacing both by $d = \max\{s, l\}$. The embedding $\imath'$ can be considered to be between the spaces $(X, d^{\alpha\beta}) \to (\mathbb{R}^m, |\cdot|^\beta)$ and $\jmath'$ between $(Y, g^{\alpha\beta}) \to (\mathbb{R}^k, |\cdot|^\alpha)$. The obvious analogues hold also for $\imath'_n$ and $\jmath'_n$. Combine these embeddings with those provided by Assouad's theorem to obtain

$$\begin{aligned} \imath_n &:= \tau_1 \circ \imath'_n : (X_n, (d_n)^{\alpha\beta}) \to \mathbb{R}^d,\ \imath := \tau_1 \circ \imath' : (X, d^{\alpha\beta}) \to \mathbb{R}^d \\ \jmath_n &:= \tau_1 \circ \jmath'_n : (X_n, (d_n)^{\alpha\beta}) \to \mathbb{R}^d,\ \jmath := \tau_1 \circ \jmath' : (Y, g^{\alpha\beta}) \to \mathbb{R}^d. \end{aligned}$$

(The reference to the metric used in $\mathbb{R}^d$ is omitted for the rest of the proof.)

To see that $\jmath_n X_n \to \jmath Y$ as subsets of $\mathbb{R}^d$ calculate for arbitrary $q \in \mathbb{R}^d$, $R > 0$

$$\begin{aligned} \sup_{z\in \jmath Y\cap B(q,R)} \operatorname{dist}(z, \jmath_n X_n) &\leq C \sup_{w\in \jmath' Y\cap B(q,R)} \operatorname{dist}(w, \jmath'_n X_n)^\alpha \overset{n\to\infty}{\longrightarrow} 0, \\ \sup_{z\in \jmath_n X_n\cap B(q,R)} \operatorname{dist}(z, \jmath Y) &\leq C \sup_{w\in \jmath'_n X_n\cap B(q,R)} \operatorname{dist}(w, \jmath' Y)^\alpha \overset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

The constant C appearing in the estimates above is the mutual bi-Lipschitz constant of all the embeddings. In a similar manner it can be seen that $\imath_n X_n \to \imath X$ as subsets of $\mathbb{R}^d$.

Let $\varphi_n := \jmath_n \circ \imath_n^{-1} : \imath_n X_n \to \jmath_n X_n$. This is a C^2-bi-Lipschitz bijection for all n. With the aid of the following Proposition, an easy consequence of (Proposition 5.1.9, [21]), it will be shown that passing to a subsequence φ_n has a limit in the sense of Definition 8.1.2.

Proposition 8.1.13. *Let $f_n : F_n \to G_n$ be a sequence of functions, where F_n is a nonempty closed subset of a complete doubling space (X, d) for each n, and G_n similarly for another complete doubling space (Y, g). Suppose the sequence (f_n) is equicontinuous and uniformly bounded on bounded sets. Then there exists a subsequence (f_{n_j}) and a mapping $f : F \to G$ with $\varnothing \neq F \subset X$ and $\varnothing \neq G \subset Y$ closed such that $f_{n_j} \to f$ in the sense of Definition 8.1.2.*

To use Proposition 8.1.13 the equicontinuity and uniform boundedness (on bounded sets) of (φ_n) needs to be verified. Equicontinuity is clear since the φ_n's are Lipschitz with a uniform constant. If $x_n \in \imath_n X_n$ and $R > 0$ then

$$\sup_{z,w \in \imath_n X_n \cap B(x_n,R)} |\varphi_n(x) - \varphi_n(y)| \leq \sup_{z,w \in \imath_n X_n \cap B(x_n,R)} C|z - w| \leq 2CR$$

so that (φ_n) is also uniformly bounded on bounded sets.

Having verified the assumptions of Proposition 8.1.13 for (φ_n) it can be concluded that there is a convergent subsequence with limit $\varphi : F \to G$. Here F and G are nonempty closed subsets of $\mathbb{R}^d$. Definition 8.1.2 requires that $\imath_n X_n \to F$ and $\jmath_n X_n \to G$. Since it is known that $\imath_n X_n \to \imath X$ and $\jmath_n X_n \to \jmath Y$ Proposition 8.1.10 implies $F = \imath X$ and $G = \jmath Y$. Now define $\phi : X \to Y$ by $\phi = \jmath^{-1} \circ \varphi \circ \imath$. Take $x, y \in X$ and $x_n, y_n \in X_n$ so that $|\imath_n x_n - \imath x| \to 0$ and similarly for the y's. From Definition 8.1.4 it follows that

$$d_n(x_n, y_n) \to d(x, y).$$

The fact that $\varphi_{n_k} \to \varphi$ for some subsequence (φ_{n_k}) implies, in particular, that $|\varphi_{n_k} \imath_{n_k} x_{n_k} - \varphi \imath x| \to 0$ as $k \to \infty$. Notice that $|\varphi_{n_k} \imath_{n_k} x_{n_k} - \varphi \imath x| = |\jmath_{n_k} x_{n_k} - \jmath \phi x|$. By applying the analogous fact for the y's and Definition 8.1.2 it can be concluded that

$$d_{n_k}(x_{n_k}, y_{n_k}) \to g(\phi x, \phi y).$$

These two facts together imply

$$g(\phi x, \phi y) = d(x, y)$$

for all $x, y \in X$.

It remains to show the bijectivity of ϕ. To this end it is sufficient to demonstrate that φ is bijective. For this consider the function sequence $f_n := \varphi_n^{-1} : \jmath_n X_n \to \imath_n X_n$. As in the case of (φ_n) it can be seen that (f_n) satisfies the assumptions of Proposition 8.1.13 and hence, after passing to a subsequence, has a limit $f : \jmath Y \to \imath X$. Now let n_k be the subscript of the mutual subsequence, so that both f and φ arise as limits of f_{n_k} and φ_{n_k}. For any $x \in \imath X$ and a sequence $x_{n_k} \in X_{n_k}$ for which $x = \lim_{k\to\infty} x_{n_k}$ one has $\varphi(x) = \lim_{k\to\infty} \varphi_{n_k}(x_{n_k})$. Then by the convergence $f_{n_k} \to f$

$$f \circ \varphi(x) = f(\varphi(x)) = \lim_{k\to\infty} f_{n_k}(\varphi_{n_k}(x_{n_k})) = \lim_{k\to\infty} x_{n_k} = x.$$

In a similar manner it is seen that $\varphi \circ f = id_Y$

□

It may be noted that the proofs of the claims above transfer directly to the case of pointed metric spaces. This is due to the fact that in that notion of convergence all the embeddings are required to respect basepoints, a feature which persists in the above discussions.

Theorem 8.1.14. *Let (X, d, p, u) be a space-function where (X, d) is a complete and doubling space and u is L-Lipschitz. For any sequence (r_n) of positive real numbers converging to zero there exists a tangent space function $(X_\infty, d_\infty, x_\infty, u_\infty) \in T(X, d, p, u)$ subordinate to a subsequence of (r_n) such that (X_∞, d_∞) is doubling with doubling constant depending only on that of (X, d) and u_∞ is L-Lipschitz.*

In the proof of this the embedding theorem of Assouad will be used.

Proof of 8.1.14. Let (X, d, p) and u satisfy the assumptions of 8.1.14. Fix some $r > 0$ and a sequence r_n of positive reals converging to zero. To prove the theorem one needs to consider the sequence (X_n, d_n, p_n, u_n) with

$$X_n = \overline{B}(p, r),\ d_n = d/r_n,\ p_n = p \text{ and } u_n(\cdot) = \frac{u(\cdot) - u(p)}{r_n}$$

and demonstrate that this has a convergent subsequence. To accomplish this fix some $0 < \alpha < 1$ and let $\imath : (X, d^\alpha) \to \mathbb{R}^m$ be a C-bi-Lipschitz embedding (assured by Assouad's embedding theorem). Normalize $\imath$ so that $\imath(p) = 0$. Then set $\imath_n = r_n^{-1}\imath$ whence $\imath_n : (X_n, (d_n)^\alpha) \to \mathbb{R}^m$ remains a C-bi-Lipschitz embedding with $\imath_n(p_n) = 0$. The sets $F_n := \imath_n X_n$ satisfy the following property: since

$$|\imath_n y| = |\imath_n p_n - \imath_n y| \le C d_n(p, y)^\alpha \le C r^\alpha$$

for $y \in B(p, r_n r) \subset B(p, r)$ (for sufficiently large n) it follows that $F_n \cap B(0, Cr^\alpha) \neq \varnothing$ for sufficiently large n. Hence according to lemma 8.1.9 there is a subsequence convergent to some closed $F \subset \mathbb{R}^m$. Denote this subsequence by the subscript k. In a similar fashion $F_k \times F_k \to F \times F$ as $k \to \infty$. [5] In any case $0 \in F$ but if, for instance, $\overline{B}(p, r)$ is a discrete set F might contain only one point.

Next consider the map $f_k(x, y) = d_k(\imath_k^{-1}x, \imath_k^{-1}y)^\alpha$ for $(x, y) \in F_k \times F_k$. It satisfies

$$|f_k(x, y) - f_k(z, w)| \le C||x - y| - |z - w|| \le C'|(x, y) - (z, w)| \text{ and} \qquad (8.1.1)$$
$$\sup_{(x,y)\in B\cap F_k} |f_k(x, y)| \le C \operatorname{diam}(B \cap F_k) \le C \operatorname{diam}(B)$$

for each bounded set $B \in \mathbb{R}^m$.

The aim is to pass somehow to the limit and consider $\lim_{k\to\infty} f_k$ which would then define a metric on $F \times F$ one could hope to be a limit of d_k^α. To accomplish this fix a countable dense subset E of $F \times F$. For each (x^j, y^j) choose a sequence $(x_k^j, y_k^j)_{k\in\mathbb{N}} \subset F_k \times F_k$ converging to (x^j, y^j). (Note that both F_k and F are subsets of $\mathbb{R}^m$.) $(f_k(x_k^1, y_k^1))$ is a bounded sequence and thus has a convergent subsetquence with limit b^1. From this extract another subsequence (labeled

[5] It is a general fact that convergence is preserved under cartesian products.

for notational reasons by the same subscripts) so that $(f_k(x_k^2, y_k^2))$ converges to some b^2 and so on. From the infinite array

$$\begin{array}{cccc} f_1(x_1^1, y_1^1) & f_2(x_2^1, y_2^1) & \dots & \to b^1 \\ f_1(x_1^2, y_1^2) & f_2(x_2^2, y_2^2) & \dots & \to b^2 \\ f_1(x_1^3, y_1^3) & f_2(x_2^3, y_2^3) & \dots & \to b^3 \\ \vdots & \ddots & \dots & \end{array}$$

choose the diagonal subsequence. Then for each j the sequence $f_k(x_k^j, y_k^j)$ where k runs over the subscripts determined by the diagonal converges to b^j. Define f on $F \times F$ as follows. Given two points $x, y \in F$ let (x^j, y^j) be a sequence in E converging to (x, y), and let $x_{k_j}^j \in F_k$ so that $|x^j - x_{k_j}^j| < 1/j$ (similarly for y). Set

$$f(x, y) = \lim_{j \to \infty} f_{k_j}(x_{k_j}^j, y_{k_j}^j)^{1/\alpha}.$$

This is well defined since if (z_j, w_j) is another sequence in E converging to (x, y) the estimate (8.1.1) above yields

$$|f_{k_j}(x_{k_j}^j, y_{k_j}^j) - f_{k_j}(z_{k_j}^j, w_{k_j}^j)| \le C'|(x_{k_j}^j - z_{k_j}^j, y_{k_j}^j - w_{k_j}^j)| \overset{j \to \infty}{\longrightarrow} 0$$

Each $f_k^{1/\alpha} = d_k(\imath_k^{-1}\cdot, \imath_k^{-1}\cdot)$ determines a metric in F_k and the properties of a metric persist under pointwise limit, hence f defines a metric on F. Moreover the uniform estimates

$$1/C|x - y| \le f_k(x, y) = d_k(\imath_k^{-1}x, \imath_k^{-1}y)^\alpha \le C|x - y|$$

also persist in the limit so that

$$1/C|x - y| \le f(x, y)^\alpha \le C|x - y| \text{ for } x, y \in F.$$

Now it is easy to show that $d_k \to f$ in the sense of 8.1.2: suppose $x, y \in F$ are given and $x_k, y_k \in X_k$ are sequences so that $|x - \imath_k x_k| \to 0$ and $|y - \imath_k y_k| \to 0$ as $k \to \infty$. For each k choose $x^{j_k} \in E \cap B(x, |x - \imath_k x_k|) \cap F_k$ and similarly for y. then

$$\begin{aligned} &\limsup_{k \to \infty} |d_k(x_k, y_k)^\alpha - f(x, y)^\alpha| = \limsup_{k \to \infty} |f_k(\imath_k x_k, \imath_k y_k) - f_k(x^{j_k}, y^{j_k})| \\ &\le C' \limsup_{k \to \infty} |(\imath_k x_k - x^{j_k}, \imath_k y_k - y^{j_k})| = 0. \end{aligned}$$

Hence the sequence $(X_k, d_k, p) \to (F, f, 0)$ in the sense of 8.1.6. By a similar argument that was was used to construct f it can be shown that, passing to yet another subsequence, the sequence $u_k : (X_k, d_k, p) \to \mathbb{R}$ converges to some $u : (F, f, 0) \to \mathbb{R}$. The essential estimate (8.1.1) follows for u_k using the fact that it is Lipschitz. Also u remains Lipschitz with the same constant by the persistence of inequalities under pointwise limits. □

Theorem 8.1.14 is a very useful one because in the context of doubling spaces it asserts the existence of tangent function-spaces at any given point. The business of the next subsection is to examine the additional properties satisfied by the tangent spaces under the hypothesis that the space supports a p-Póincare inequality for some $p \ge 1$ and link these to the differential structure 8.0.5 defined in the previous section.

8.2 Tangent functions

In this section (X, d, μ) will always assumed to be a complete doubling metric space.

Definition 8.2.1. *A Lipschitz function $u : X \to \mathbb{R}$ is said to be K-quasilinear if for every $x \in X$ and $0 < r \leq diam(X)$*

$$\mathrm{LIP}(u) \leq K \operatorname*{var}_{x,r} u$$

where

$$\operatorname*{var}_{x,r} u = L_r u(x) = \sup_{y \in B(x,r)} \frac{|u(x) - u(y)|}{r}.$$

It can immediately be seen that K, if it exists, must be at least one since $\operatorname*{var}_{x,r} u \leq \mathrm{Lip}(u)$ holds for any $u \in \mathrm{LIP}(X)$. The motivation behind the name in the definition is that if X is a normed space, for instance $X = \mathbb{R}^n$, then any linear map $u : X \to \mathbb{R}$ is 1-quasilinear.

The following Proposition constitutes the first of several upcoming finite - dimensionality results.

Proposition 8.2.2. *Suppose $V \subset \mathrm{LIP}(X)$ is a vector space consisting of K-quasilinear functions for a given K, and a point $x_0 \in X$ is fixed. If V has the property that every $u \in V$ satisfies $u(x_0) = 0$, then V is finite dimensional with dimension bounded above by a constant depending only on K and the doubling constant for μ.*

Proof. By [30] a complete doubling space carries a doubling Borel regular measure μ on it the doubling constant of which depends only on that of the space. Throughout the proof μ will stand for such a fixed measure on X.

Suppose $x_0 \in X$ is as in the claim and $1 > t > 0$. Construct a maximal sequence $x_1, \ldots, x_n \in B(x_0, 1) := B$ having the property that

$$x_{n+1} \in B \setminus \bigcup_{i=1}^{n} B(x_i, t) \text{ and } B(x_{n+1}, t/2) \cap B(x_i, t/2) = \varnothing \text{ for all } i \leq n$$

akin to the proof of Proposition 2.16. This is finite, and using Proposition 2.14 n can be estimated from above by

$$n \leq C^4 t^{-s},$$

C being the doubling constant of the measure and $s = \log_2 C$. The balls $B_i = B(x_i, t)$ cover B (for all this see proof of 2.16). Now for any $i = 1, \ldots, n$ suppose M_i is the set of indices j for which the balls B_j and B_i intersect. For any $j \in M_i$ it follows that $d(x_i, x_j) \leq 2t$, hence $B(x_i, 3t)$ contains each ball B_j. Since the balls $1/2B_j$, $j \in M_i$ are disjoint one has

$$1 \geq \frac{\mu(\bigcup_{j \in M_i} 1/2B_j)}{\mu(3B_i)} = \sum_{j \in M_i} \frac{\mu(1/2B_j)}{\mu(3B_i)} \geq |M_i| \cdot 4^{-s} \left(\frac{t/2}{3t}\right)^s.$$

Thus for each i the quantity $|M_i|$ has an upper bound 24^s which depends only on the doubling constant of the measure. Let $n =: n(t)$ and M be the largest integers smaller than C^4t^{-s} and 24^s, respectively. Define $f_t : V \to \mathbb{R}^{n(t)}$,

$$f_t(u) = (\mu(B_1)u_{B_1}, \ldots, \mu(B_{n(t)})u_{B_{n(t)}}).$$

Clearly this is a linear map $V \to \mathbb{R}^{n(t)}$. To prove the claim it suffices to show that there exists some t (depending on K and the doubling constant) so that f_t is injective, since from that it is clear that any V satisfying the conditions in the claim can have dimension at most $N = n(t) = n(K, C)$.

For any $u \in V$ there is a point $x \in B(x_0, 1/2)$ so that
$\mathrm{LIP}(u) \leq K\dfrac{|u(x) - u(x_0)|}{1/2} = 2K|u(x)|$. Consequently for any $y \in B(x, 1/(3K))$ the estimate

$$\begin{aligned}|u(y)| &= |u(x) - (u(x) - u(y))| \geq |u(x)| - |u(x) - u(y)| \\ &\geq \frac{\mathrm{LIP}(u)}{2K} - \mathrm{LIP}(u)d(x,y) \geq \frac{\mathrm{LIP}(u)}{6K}\end{aligned}$$

holds. Therefore

$$⨍_{B(x,1/(3K))} |u| \mathrm{d}\mu \geq \frac{\mathrm{LIP}(u)}{6K}.$$

The inclusion $B(x, 1/(3K)) \subset B(x_0, 1/2 + 1/(3K)) \subset B$ and the doubling property of μ imply that there is a constant $L = L(C, K)$ so that

$$\frac{\mathrm{LIP}(u)}{6K} \leq ⨍_{B(x,1/(3K))} |u| \mathrm{d}\mu \leq L ⨍_B |u| \mathrm{d}\mu. \tag{8.2.1}$$

Now estimate

$$\int_B |u| \mathrm{d}\mu \leq \sum_{i=1}^{n} \int_{B_i} |u| \mathrm{d}\mu \leq \sum_{i=1}^{n} \int_{B_i} |u - u_{B_i}| \mathrm{d}\mu + \sum_{i=1}^{n} \mu(B_i)|u_{B_i}|, \tag{8.2.2}$$

and for each term in the first sum

$$\begin{aligned}\int_{B_i} |u - u_{B_i}| \mathrm{d}\mu &\leq \int_{B_i} ⨍_{B_i} |u(x) - u(y)| \mathrm{d}\mu(y) \mathrm{d}\mu(x) \\ &\leq \int_{B_i} ⨍_{B_i} \mathrm{LIP}(u) d(x,y) \mathrm{d}\mu(y) \mathrm{d}\mu(x) \leq 2t \, \mathrm{LIP}(u)\mu(B_i).\end{aligned}$$

Hence

$$\begin{aligned}&\sum_{i=1}^{n} \int_{B_i} |u - u_{B_i}| \mathrm{d}\mu \leq 2t \, \mathrm{LIP}(u) \sum_{i=1}^{n} \mu(B_i) \leq 2t \, \mathrm{LIP}(u) M \mu(\bigcup_{i=1}^{n} B_i) \\ &\leq 2tM \, \mathrm{LIP}(u)\mu(2B) \leq 2tMC \, \mathrm{LIP}(u)\mu(B) \leq 12tMCKL \int_B |u| \mathrm{d}\mu. \end{aligned} \tag{8.2.3}$$

The last inequality is an application of (8.2.1) and the second one is a consequence of the fact that each ball B_i intersects at most M of the balls in the covering $B_1, \dots, B_{n(t)}$. Now choose $t = \dfrac{1}{24MCKL} < 1$ and insert (8.2.3) to (8.2.2) which then becomes

$$\int_B |u| \mathrm{d}\mu \leq 1/2 \int_B |u| \mathrm{d}\mu + |f_t(u)|_1,$$

where $|\cdot|_1$ denotes the 1-norm in $\mathbb{R}^{n(t)}$. Note also that t and thus n depends only on K and the doubling constant of the measure (which in turn depends on the doubling constant of the space).

The last inequality effectively proves the injectivity of f_t: if $f_t(u) = 0$ then

$$\int_B |u| \mathrm{d}\mu \leq 0$$

implying that $u = 0$ on B but then the K-quasiconvexity of u implies that $\mathrm{LIP}(u) \leq K \operatorname*{var}_{x_0, 1/2} u = 0$ and hence $u = 0$. □

Proposition 8.2.3. *Let (X, d, μ) be a locally compact doubling metric measure space and let $u \in \mathrm{LIP}(X)$. Suppose (r_n) is a sequence of positive real numbers converging to zero. Then for almost every $x \in X$ every tangent spacefunction $(X_\infty, d_\infty, x_\infty, u_\infty) \in T(X, d, x, u)$ which is subordinate to a subsequence of (r_n) satisfies*

$$\begin{aligned} &u_\infty(x_\infty) = 0 \textit{ and} \\ &\operatorname{lip} u(x) \leq \operatorname*{var}_{y,s} u_\infty \leq \mathrm{LIP}(u_\infty) \leq \operatorname{Lip} u(x) \end{aligned}$$

for every $y \in X_\infty$ and $s > 0$.

Proof. Throughout the proof the notation of Definition 8.1.7 will be used.

Let r_n be any sequence converging to zero. X can be assumed to have finite measure since it can be split into countably many sets of finite measure. Then for every n there exists, by Egoroff's and Luzin's theorems, a measurable set A_n so that $\mu(X \setminus A_n) < 2^{-n}$ and the sequences of functions

$$L_n := \sup_{r < r_n} \sup_{d(\cdot, y) < r} \frac{|u(\cdot) - u(y)|}{d(\cdot, y)} \text{ and } l_n := \inf_{r < r_n} \sup_{d(\cdot, y) < r} \frac{|u(\cdot) - u(y)|}{d(\cdot, y)}$$

converge to $\operatorname{Lip} u$ and $\operatorname{lip} u$ uniformly on A_n, respectively, and both $\operatorname{Lip} u$ and $\operatorname{lip} u$ are continuous on A_n. Let $A = \bigcup_{n=1}^{\infty} A_n$ whence $\mu(X \setminus A) = 0$.

The first task is to prove that for almost every $x \in A_n$, for any point y from $(X_\infty, d_\infty, x_\infty, u_\infty) \in T(X, d, x, u)$ which is subordinate to a subsequence of (r_n) (relabeled (r_k)) there is a sequence $(y_k) \subset K \cap A_n$ so that $|\imath_k y_k - \imath y| \to 0$. Here $K \subset X$ is a compact set containing a neighbourhood of x for which $(K, d/r_k, x, u_k) =: (X_k, d_k, x_k, u_k) \to (X_\infty, d_\infty, x_\infty, u_\infty)$ and again $u_k(\cdot) = (u(\cdot) - u(x))/r_k$.

This is in fact true of every density point y of $K \cap A_n$: by lemma 8.1.11 there is a sequence $y_k \in K$ for which $|\imath_k y_k - \imath y| \to 0$. In particular $d(y_k, x)^\alpha \leq C r_k |\imath_k y_k - \imath y| \leq C r_k$. Suppose there doesn't exist a sequence $z_k \in A_n \cap K$

with the desired property. Then there is some $t > 0$ and a sequence k_m so that $B(y_{k_m}, tr_{k_m}) \cap A_n \cap K = \varnothing$ for all m.[6] Indeed, were this not the case then for every $t > 0$ the inequality $d(A_n \cap K, y_k)^\alpha \leq tr_k$ would hold for sufficiently large k, leading to

$$\limsup_{k\to\infty} \operatorname{dist}(\imath_k(A_n \cap K), \imath_k y_k) \leq C \limsup_{k\to\infty} d_k(A_n \cap K, y_k)^\alpha < Ct$$

for every $t > 0$. Hence there would exist a sequence $a_k \in A_n \cap K$ satisfying $\rho(\imath_k a_k, \imath y) \to 0$.

However employing the inclusion $B(y_{k_m}, tr_{k_m}) \subset B(x, (t+C)r_{k_m})$ and Proposition 2.14 the following contradictory estimate can be deduced

$$\frac{\mu(B(x,(t+C)r_{k_m}) \setminus (A_n \cap K))}{\mu(B(x,(t+C)r_{k_m}))} \geq \frac{\mu(B(y_{k_m}, tr_{k_m}))}{\mu(B(x,(t+C)r_{k_m}))} \geq 4^{-s}\left(\frac{t}{t+C}\right)^s > 0.^7$$

For the rest of the proof fix one density point of A and A_n and a tangent space of it, subordinate to a henceforth fixed subsequence r_k of r_n. To prove the last inequality it suffices to show that if $y, z \in X_\infty, y \neq z$ are arbitrary then

$$\frac{|u_\infty(y) - u_\infty(z)|}{d_\infty(y,z)} \leq \operatorname{Lip} u(x).$$

Let $y_k, z_k \in A_n \cap K$ be sequences satisfying $|\imath_k y_k - \imath y| \to 0$ and likewise for z_k and z. This is possible by lemma 8.1.11 and the above discussion. An easy computation yields

$$\frac{|u(y_k) - u(z_k)|}{d(y_k, z_k)} = \frac{|u_k(y_k) - u_k(z_k)|}{d_k(y_k, z_k)} \to \frac{|u_\infty(y) - u_\infty(z)|}{d_\infty(y,z)}.$$

The convergence here is assured by the assumption that the functions d_k and u_k converge to d_∞ and u_∞, respectively. Hence, having fixed y and z, it suffices to prove that

$$\limsup_{k\to\infty} \frac{|u(y_k) - u(z_k)|}{d(y_k, z_k)} \leq \operatorname{Lip} u(x).$$

Now $d(z_k, y_k) = r_k d_k(z_k, y_k) \leq Cr_k$. Fix some $\varepsilon > 0$. Then there is k_1 so that

$$\sup_{r<r_k} \sup_{d(w,w')<r} \frac{|u(w) - u(w')|}{d(w,w')} < \varepsilon + \operatorname{Lip} u(w) \text{ for every } w \in A_n$$

whenever $k \geq k_1$. There also exists k_2 so that $d(y_k, z_k) < r_{k_1}$ whenever $k > k_2$. For all such k it follows that

$$\frac{|u(y_k) - u(z_k)|}{d(y_k, z_k)} \leq \sup_{r<r_{k_1}} \sup_{d(y_k,w')<r} \frac{|u(y_k) - u(w')|}{d(y_k, w')} < \varepsilon + \operatorname{Lip} u(y_k)$$

yielding

$$\limsup_{k\to\infty} \frac{|u(y_k) - u(z_k)|}{d(y_k, z_k)} \leq \varepsilon + \lim_{k\to\infty} \operatorname{Lip} u(y_k) = \varepsilon + \operatorname{Lip} u(x).$$

Since this inequality holds for arbitrary ε the latter inequality is obtained.

For the first inequality in the claim suppose it was known that given $y \in X_\infty$ and $s > 0$ the following holds true.

[6] $B(y_{k_m}, tr_{k_m}) \subset X$.

[7] s denotes the homogeneity exponent of μ, see Definition 2.15 and Proposition 2.14.

Lemma 8.2.4. *There is a sequence* (y_k) *with* $y_k \in K \cap A_n$ *so that*

- $|\imath_k y_k - \imath y| \overset{k\to\infty}{\longrightarrow} 0$ *and*
- $\operatorname{lip} u(x) \leq \liminf_{k\to\infty} \sup_{d(z,y_k)<sr_k} \frac{|u(z)-u(y_k)|}{sr_k}$.

Then for arbitrary $\varepsilon > 0$ and every $n \in \mathbb{N}$ let $z_n \in B(y_n, sr_n)$ so that

$$\frac{|u(z_n)-u(y_n)|}{sr_n} + \varepsilon > \sup_{d(z,y_n)<sr_n} \frac{|u(z)-u(y_n)|}{sr_n}.$$

Since $d_n(y_n, z_n) < s$ it follows that $|\imath_n z_n - \imath_n y_n| \leq Cs^\alpha$ and consequently that $\imath_n z_n \in B(\imath y, R) \subset \mathbb{R}^m$ for some R. By the local compactness of $\mathbb{R}^m$ there is a subsequence and $z' \in B(\imath y, R)$ so that $|\imath_n z_n - \imath z'| \to 0$. According to Lemma 8.1.8 there exists some $z \in X_\infty$ such that $z' = \imath z$. Further $d_\infty(z,y) = \lim_{n\to\infty} d_n(z_n, y_n) \leq s$. Now

$$\frac{|u(z_n)-u(y_n)|}{sr_n} = \frac{|u_n(z_n)-u_n(y_n)|}{s} \overset{n\to\infty}{\to} \frac{|u_\infty(z)-u_\infty(y)|}{s} \leq \operatorname*{var}_{y,s} u_\infty.$$

This inequality, Lemma 8.2.4 and the choice of z_n together yield

$$\operatorname{lip} u(x) \leq \liminf_{n\to\infty} \sup_{d(z,y_n)<sr_n} \frac{|u(z)-u(y_n)|}{sr_n} \leq$$

$$\varepsilon + \lim_{n\to\infty} \frac{|u(z_n)-u(y_n)|}{sr_n} \leq \varepsilon + \operatorname*{var}_{y,s} u_\infty$$

which implies the desired inequality since ε is arbitrary. Thus the proof of this Proposition is reduced to that of Lemma 8.2.4 □

Proof of 8.2.4. Let $y \in X_\infty$, $s > 0$ be given. Let (y_k) be a sequence with the properties $y_k \in K \cap A_n$ and $|\imath_k y_k - \imath y| \to 0$. Then $d(x, y_k)^\alpha = r_k^\alpha d_k(x, y_k)^\alpha \leq Cr_k^\alpha |\imath_k x - \imath_k y_k| \leq Cr_k^\alpha$ for large enough k.

Fix some $\varepsilon > 0$. Then there exists some k_1 so that

$$\inf_{r<r_k} \sup_{d(w,z)<r} \frac{|u(w)-u(z)|}{r} + \varepsilon > \operatorname{lip} u(w) \text{ for all } w \in A_n$$

whenever $k \geq k_1$. Choose k_2 so that $sr_k < r_{k_1}$ whenever $k > k_2$. For such k

$$\operatorname{lip} u(y_k) < \varepsilon + \inf_{r<r_k} \sup_{d(y_k,z)<r} \frac{|u(y_k)-u(z)|}{r} \leq \varepsilon + \inf_{k>k_2} \sup_{d(y_k,z)<sr_k} \frac{|u(y_k)-u(z)|}{sr_k}$$

whence

$$\operatorname{lip} u(x) = \lim_{k\to\infty} \operatorname{lip} u(y_k) \leq \varepsilon + \liminf_{k\to\infty} \sup_{d(y_k,z)<sr_k} \frac{|u(y_k)-u(z)|}{sr_k}$$

for arbitrary $\varepsilon > 0$. This proves the claim. □

The second finite dimensionality result already presents a common tangent space for all tangent functions of certain type.

Proposition 8.2.5. *Let (X, d, μ) be a complete and doubling metric space and $\mathbf{u} = (u_1, \dots, u_n) : X \to \mathbb{R}^n$ so that $u_i \in \mathrm{LIP}(X)$ for every $1 \le i \le n$. Further let (r_n) be any sequence of positive reals converging to zero. Suppose there is a constant $K > 0$ so that if $\lambda \in \mathbb{R}^n$ then*

$$\mathrm{Lip}(\lambda \cdot \mathbf{u})(x) \le K \,\mathrm{lip}(\lambda \cdot \mathbf{u})(x) \text{ for almost every } x \in X.$$

Then for almost every $x \in X$ there is a tangent space $(X_\infty, d_\infty, x_\infty)$ of (X, d, x) subordinate to a subsequence of (r_n) with the following properties: for each $\lambda \in \mathbb{R}^n$ there is a tangent function $u_{\lambda,x} : (X_\infty, d_\infty, x_\infty) \to \mathbb{R}$ of $\lambda \cdot \mathbf{u}$ for which

i) $u_{\lambda,x}(x_\infty) = 0$

ii) $u_{\lambda,x}$ *is K-quasilinear.*

For $x \in X$ for which this space exists the dimension of the vector space $V_x := \{u_{\lambda,x} : \lambda \in \mathbb{R}^n\}$ has an upper bound. That is, there exists a constant $N \in \mathbb{N}$ depending only on K and the doubling constant of μ so that $\dim V_x \le N$.

Proof. Let $\mathbf{u}$, K and (r_k) be as in the assumptions. Denote by $\{e_i\}_{i=1}^n$ the standard basis of $\mathbb{R}^n$. For almost every $x \in X$ select a subsequence $(r^0_{x,k})_k$ of (r_k) so that the claim of proposition 8.2.3 is satisfied for $e_1 \cdot \mathbf{u}$. Then, by theorem 8.1.14 select a subsequence $(r^1_{x,k})_k$ of $(r^0_{x,k})_k$ so that there exists a doubling tangent space (X_x, d_x, x_∞) of (X, d, x) and a tangent function $u_{1,x}$ of $e_1 \cdot \mathbf{u}$ subordinate to $(r^0_{x,k})_k$. Note that proposition 8.2.3 and the assumptions of this proposition guarantee for almost every $x \in X$

$$\operatorname*{var}_{y,s} u_{1,x} \ge \mathrm{lip}(\lambda \cdot \mathbf{u})(x) \ge 1/K \,\mathrm{Lip}(\lambda \cdot \mathbf{u})(x) \ge 1/K \,\mathrm{LIP}(u_{1,x})$$

for *every* $y \in X_x$ and $s > 0$. Thus $u_{1,x}$ is K-quasilinear. From $(r^1_{x,k})_k$ choose another subsequence $(r^{1,0}_{x,k})_k$ so that the claims of proposition 8.2.3 hold for the function $e_2 \cdot \mathbf{u}$ and from this yet another (labeled $(r^2_{x,k})_k$), assured by theorem 8.1.14 so that there is a tangent space function $(X'_x, d'_x, x'_\infty, u_{2,x}) \in T(X, d, x, e_2 \cdot \mathbf{u})$ subordinate to $(r^2_{x,k})_k$. Again $u_{2,x}$ is K-quasilinear. Now the pointed metric spaces (X_x, d_x, x_∞) and (X'_x, d'_x, x'_∞) are both subordinate to $(r^2_{x,k})_k$. Hence, as limits of the respective sequences they are isometrically equivalent and the domain of $u_{2,x}$ can be taken to be (X_x, d_x, x_∞). $u_{2,x}$ then remains K-quasilinear. After n repeated applications of this procedure the result is a subsequence $(r^n_{x,k})_k$, a tangent space (X_x, d_x, x_∞) of (X, d, x) and tangent functions $u_{i,x} : X_x \to \mathbb{R}$ of $e_i \cdot \mathbf{u}$ $(1 \le i \le n)$ – all subordinate to $(r^n_{x,k})_k$. Further the functions $u_{i,x}$ are all K-quasilinear.

Suppose some $x \in X$ which admits the above discussion is fixed. For any $\lambda \in \mathbb{R}^n$ the function $\lambda \cdot \mathbf{u}$ can be written as $\lambda \cdot \mathbf{u} = \sum_{i=1}^{n} \lambda_i e_i \cdot \mathbf{u}$. The sum $\sum_{i=1}^{n} \lambda_i u_{i,x} =: u_{\lambda,x}$, defined on (X_x, d_x, x_∞) is a tangent function for $\lambda \cdot \mathbf{u}$ (subordinate to $(r^n_{x,k})_k$). This is seen easily from the definition of tangent space functions: If f_∞ and g_∞ are tangent functions of f and g respectively, subordinate to the same sequence and $(X_\infty, d_\infty, x_\infty)$ – a tangent space of (X, d, x) – their common domain of definition then by considering the limits $f_n + g_n$ on (X_n, d_n, x_n) it is seen that $f_\infty + g_\infty = \lim_{n\to\infty}(f_n + g_n) = \lim_{n\to\infty} f_n + \lim_{n\to\infty} g_n$ is a tangent function for $f + g$.

Since $u_{i,x}(x_\infty) = 0$ for all $i \in \{1, \dots, n\}$ then by the definition of the tangent function the same holds for $u_{\lambda,x}$, $\lambda \in \mathbb{R}^n$. Also, since $u_{\lambda,x}$ is a tangent function of $\lambda \cdot \mathbf{u}$ and the condition $\mathrm{Lip}(\lambda \cdot \mathbf{u})(x) \le K \,\mathrm{lip}(\lambda \cdot \mathbf{u})(x)$ is satisfied (for almost every x) it follows together with Proposition 8.2.3 that

$$\mathrm{LIP}(u_{\lambda,x}) \le \mathrm{Lip}(\lambda \cdot \mathbf{u})(x) \le K \,\mathrm{lip}(\lambda \cdot \mathbf{u})(x) \le K \underset{(y,s)}{\mathrm{var}}\, u_{\lambda,x},$$

for all $y \in (X_x, d_x, x_\infty)$, $s > 0$ – that is, $u_{\lambda,x}$ is K-quasilinear.

Finally that the dimension of the space $V := \{u_{\lambda,x} : \lambda \in \mathbb{R}^n\}$ has an upper bound follows directly from proposition 8.2.2 which, indeed, provides an upper bound for vector spaces such as the above depending only on K and the doubling constant of X_∞ which, in turn, depends only on the doubling constant of X (as asserted by Theorem 8.1.14). □

The obtained upper bound does not á priori constrict the degree of freedom n as such, since the vector space V_x could have dimension strictly less than n. (It is however at most n.) The following proposition states that under one more hypothesis the dimension of the vector space V_x is preserved. This will be crucial in the follow-up.

Proposition 8.2.6. *If, in the situation of the previous theorem there is a measurable set $A \subset X$ with positive measure and a constant $\delta > 0$ so that for each $\lambda \in \mathbb{R}^n$*

$$\mathrm{Lip}(\lambda \cdot \mathbf{u})(a) \ge \delta|\lambda| \text{ for almost every } a \in A, \tag{8.2.4}$$

then the dimension of $V_a =: \{u_{\lambda,a} : \lambda \in \mathbb{R}^n\}$ equals $\dim V_a = n$ for almost every $a \in A$.

Proof. If $\lambda \in \mathbb{R}^n$ is fixed then there is a set $A_\lambda \subset A$ so that $\mu(A \setminus A_\lambda) = 0$ and (8.2.4) holds for *every* $a \in A_\lambda$. However if some $a \in A$ is fixed then it is *not* true that for every $\lambda \in \mathbb{R}^n$ equation (8.2.4) should hold. It would be true is a were taken from the intersection of all A_λ, $\lambda \in \mathbb{R}^n$ but the problem here is that the intersection is uncountable. To circumvent this problem the intersection will be taken over Λ, a countable dense subset of $\mathbb{R}^n$ which from now forth is fixed.

Denote $A_\Lambda = \bigcap_{\lambda\in\Lambda} A_\lambda$. Then if $a \in A_\Lambda$ is fixed equation (8.2.4) holds for every $\lambda \in \Lambda$. Let B be the intersection of A_Λ with the set of points where the claim of proposition 8.2.3 is valid for every $\lambda \in \Lambda$ and with the set of points where V_a can be formed. Since all these requirements are valid almost everywhere the set B still satisfies $\mu(A \setminus B) = 0$. Now fix $a \in B$. For an arbitrary vector $\lambda \in \mathbb{R}^n$ let λ_k be a sequence in Λ converging to λ. Then

$$|\mathrm{Lip}(\lambda_k \cdot \mathbf{u})(a) - \mathrm{Lip}(\lambda \cdot \mathbf{u})(a)| \le \mathrm{Lip}((\lambda_k - \lambda) \cdot \mathbf{u})(a) \le \sum_{i=1}^{n} |\lambda_k^i - \lambda^i| \,\mathrm{Lip}\, u_i(a) \overset{k\to\infty}{\longrightarrow} 0$$

which implies

$$\mathrm{Lip}(\lambda \cdot \mathbf{u})(a) = \lim_{k\to\infty} \mathrm{Lip}(\lambda_k \cdot \mathbf{u})(a) \ge \limsup_{k\to\infty} \delta|\lambda_k| = \delta|\lambda|.$$

in other words for $a \in B$ equation (8.2.4) holds for every $\lambda \in \mathbb{R}^n$.

Now define $L := \lambda \mapsto u_{\lambda,a} : \mathbb{R}^n \to V_a$. It will be shown that this is a bijective linear map; indeed linearity and surjectivity is quite clear so it suffices to show the injectivity of L. For this suppose $0 = L(\lambda) = u_{\lambda,a}$. Choose a sequence $(\lambda_k) \subset \Lambda$ converging to λ. From Proposition 8.2.3 one has

$$\begin{aligned}&\limsup_{k\to\infty} \operatorname{LIP}(u_{\lambda_k,a}) \geq \limsup_{k\to\infty} \operatorname{lip}(\lambda_k \cdot \mathbf{u})(a) \geq \limsup_{k\to\infty} \operatorname{Lip}(\lambda_k \cdot \mathbf{u})(a)/K \\ &\geq \limsup_{k\to\infty} \delta|\lambda_k|/K \geq \delta|\lambda|/K. \qquad (8.2.5)\end{aligned}$$

On the other hand, from the assumptions and by Proposition 8.2.3

$$\begin{aligned}&\limsup_{k\to\infty} \operatorname{LIP}(u_{\lambda_k,a}) \leq \limsup_{k\to\infty} \operatorname{Lip}(\lambda_k \cdot \mathbf{u})(a) \leq K \limsup_{k\to\infty} \operatorname{lip}(\lambda_k \cdot \mathbf{u})(a) \\ &\leq K \limsup_{k\to\infty} \operatorname*{var}_{y,s}\, u_{\lambda_k,a} \qquad (8.2.6)\end{aligned}$$

for every $y \in X_\infty$ and $s > 0$. Now from the proof of proposition 8.2.5 it can be seen that $u_{\lambda,a} = \lambda \cdot (u_{1,a}, \ldots, u_{n,a})$ where $u_{i,a}$ is the tangent function of $e_i \cdot \mathbf{u}$. Therefore $u_{\lambda_k,a} \to u_{\lambda,a}$ uniformly on compact subsets of X_∞. In particular it is is easy to check that

$$\limsup_{k\to\infty} \operatorname*{var}_{y,s}\, u_{\lambda_k,a} \leq \operatorname*{var}_{y,s}\, u_{\lambda,a}$$

since $B_\infty(y,s) \subset X_\infty$ is compact. Now $u_{\lambda,a} = 0$ implies $\operatorname*{var}_{y,s}\, u_{\lambda,a} = 0$ and consequently $|\lambda| = 0$ by (8.2.5) and (8.2.6) whence the injectivity of L follows. Hence in particular the dimension of V_a equals n. □

Propositions 8.2.5 and 8.2.6 immediately imply the following finite-dimensionality result.

Corollary 8.2.7. *Let (X,d,μ) be a complete and doubling metric space and $\mathbf{u} = (u_1, \ldots, u_n) : X \to \mathbb{R}^n$ so that $u_i \in \operatorname{LIP}(X)$ for every $1 \leq i \leq n$. Suppose there exists a constant $K > 0$ so that if $\lambda \in \mathbb{R}^n$ then*

$$\operatorname{Lip}(\lambda \cdot \mathbf{u})(x) \leq K \operatorname{lip}(\lambda \cdot \mathbf{u})(x) \text{ for almost every } x \in X,$$

and there is a measurable set $A \subset X$ with positive measure and a constant $\delta > 0$ so that for each $\lambda \in \mathbb{R}^n$

$$\operatorname{Lip}(\lambda \cdot \mathbf{u})(a) \geq \delta|\lambda| \text{ for almost every } a \in A.$$

Then n cannot exceed a constant N depending only on K and the doubling constant of X.

8.3 The differential structure

The main result of this subsection will be the following theorem

Theorem 8.3.1. *Let (X,d,μ) be a locally compact metric space and μ a doubling measure. If there exists some K so that*

$$\operatorname{Lip} u(x) \leq K \operatorname{lip} u(x) \ \mu\text{-almost everywhere}$$

for all $u \in \mathrm{LIP}(X)$, then there is a natural number N (depending only on K and the doubling constant of the measure) such that for any μ-measurable A with $\mu(A) > 0$ there exists a μ-measurable $V \subset A$ with $\mu(V) > 0$, a natural number $0 \leq n \leq N$, a positive number $\delta > 0$ and a function

$$\mathbf{f} = (f_1, \ldots, f_n) : X \to \mathbb{R}^n$$

such that $f_i \in \mathrm{LIP}(X)$ for $1 \leq i \leq n$ and further $\mathbf{f}$ has the following property: for any $u \in \mathrm{LIP}(X)$ there is a unique (up to a set of measure zero) measurable function $\mathrm{d}u : V \to \mathbb{R}^n$ which satisfies

$$\lim_{y \to x} \frac{|u(y) - u(x) - \mathrm{d}u(x) \cdot (\mathbf{f}(x) - \mathbf{f}(y))|}{d(y, x)} = 0 \text{ for every } x \in V.$$

Further for almost all $x \in V$ the inequality

$$\mathrm{Lip}(\lambda \cdot \mathbf{f})(x) \geq \delta |\lambda|$$

holds for every $\lambda \in \mathbb{R}^n$.

Before starting with the proof of Theorem 8.3.1, however, it will be shown how Theorem 8.0.5 follows from 8.3.1. To this purpose the following Lemma will be employed

Lemma 8.3.2. *Suppose (X, μ) be a σ-finite measure space and P is some property defined for the measurable sets of X (i.e. a measurable set $A \subset X$ either does or does not have the property P) which obeys the following: Every measurable set $A \subset X$ with positive measure contains a subset $V \subset A$ of positive measure such that V has the property P. Then there is a countable disjoint decomposition*

$$X = Z \cup \bigcup_i V_i$$

of X so that $\mu(Z) = 0$, $\mu(V_i) > 0$ for all $i \in \mathbb{N}$ and each V_i has the property P.

Proof. By the σ-finiteness of X there is a countable collection $U_1, U_2, \ldots$ of measurable subsets of X such that $X = \bigcup_k U_k$ and $\mu(U_k) < \infty$ for every $k \in \mathbb{N}$. It can also be assumed that $\mu(U_k) > 0$ for every $k \in \mathbb{N}$. Further the sets can be taken mutually disjoint. This sequence can also be finite. Fix a $k \in \mathbb{N}$.

Let Ω be the set of all sequences $(V_i)_{i<\alpha}$, $\alpha < \omega_1$ [8] of measurable subsets of U_k, indexed by countable ordinals, that satisfy $\mu(V_i) > 0$, V_i has the property P for all i and $V_{j+1} \subset U_k \setminus \bigcup_{i=1}^{j} V_i$. Introduce an ordering $\leq$ in Ω by defining

$$(V_i)_{i<\alpha} \leq (W_i)_{i<\beta}$$

if $\alpha \leq \beta$ and $V_i = W_i$ for all $i = 1, \ldots, \alpha$. Here α and β are allowed to range over the countable ordinals. For any chain of sequences A of Ω the sequence which contains all the members of all the sequences of A is an upper bound for A: if it weren't, then we would obtain an uncountable sequence of mutually

[8] ω_1 denotes the first uncountable ordinal. For a study of ordinals see [6]

disjoint measurable subsets with *positive* measure. This is impossible. Thus an application of Zorn's lemma yields a maximal sequence $(V_i)_{i<\alpha}$ in Ω. Then $\mu(U_k \setminus \bigcup_i V_i) = 0$ because otherwise there would exist a set $V' \subset U_k \setminus \bigcup_{i<\alpha} V_i$ with positive measure having the property P. This would contradict the maximality of $(V_i)_{i<\alpha}$ (since $\alpha+1 > \alpha$ for any ordinal α). Thus a mutually disjoint sequence of positive measured sets with the property P is obtained, such that they cover U_k apart from a set of measure zero.

By doing this to every k two sequences $(V_i)_{i=1}^{\infty}$ and (Z_k) are obtained, the first all having positive measure, being mutually disjoint having the property P and the second satisfying $\mu(Z_k) = 0$ for all k, such that

$$X = Z \cup \bigcup_i V_i$$

where $Z = \bigcup_k Z_k$ is a set of measure zero. This completes the proof. □

Theorem 8.3.1 then states that if the property P is taken to be the existence of $n, N \in \mathbb{N}$, $\overline{f}$ and $\mathrm{d}u$ as in 8.3.1 then for any metric space satisfying certain conditions the property P obeys the assumption made in 8.3.2.

It can therefore be deduced that if theorem 8.3.1 holds true and (X, d, μ) is a locally compact doubling metric measure space then there exists a natural number N and a countable collection (V_i) of sets of positive measure and positive numbers δ_i so that for each i there is a natural number n_i, functions $\overline{f}_i : X \to \mathbb{R}^{n_i}$ – that is a countable collection $(V_i, \overline{f}_i)$ of *pairs* – where $\overline{f}_i = (f_i^1, \dots, f_i^{n_i})$ and $f_i^j \in \mathrm{LIP}(X)$ so that for each $u \in \mathrm{LIP}(X)$ there is a function $\mathrm{d}_i u : V_i \to \mathbb{R}^{n_i}$ (for each $i \in \mathbb{N}$), for which

$$\lim_{\substack{y \to x \\ y \neq x}} \frac{|u(y) - u(x) - \mathrm{d}_i u(x) \cdot (\overline{f}_i(x) - \overline{f}_i(y))|}{d(y, x)} = 0 \text{ and} \qquad (8.3.1)$$
$$\mathrm{Lip}(\lambda \cdot \overline{f}_i)(x) \geq \delta_i |\lambda| \text{ for every } \lambda \in \mathbb{R}^{n_i}$$

almost everywhere. This sequence of sets *almost* covers X – in other words there is a set Z of measure zero so that

$$X = Z \cup \bigcup_i V_i.$$

Since (8.3.1) is required to hold almost everywhere the null set Z can be absorbed into one of the sets V_i. Hence to prove the existence of the differential structure 8.0.5 it suffices to prove 8.3.1. A few more definitions used in the proof will now be presented.

Definition 8.3.3. *For any point x in a metric space (X, d) the seminorm $|\cdot|_x$ in the set* $\mathrm{LIP}(X)$ *is defined by*

$$|u|_x = \mathrm{Lip}\, u(x).$$

Of course one has to prove that $|\cdot|_x$ is, indeed, a seminorm.

Lemma 8.3.4. *For any $x \in X$, $|\cdot|_x$ is a seminorm.*

Proof. Let $u, v \in \mathrm{LIP}(X)$ and $t \in \mathbb{R}$. Then

$$\mathrm{Lip}(tu)(x) = \limsup_{r\to 0} \sup_{d(x,y)<r} \frac{|tu(x) - tu(y)|}{r} = |t| \limsup_{r\to 0} \sup_{d(x,y)<r} \frac{|u(x) - u(y)|}{r}$$

and similarly

$$\mathrm{Lip}(u+v)(x) \le \limsup_{r\to 0} \sup_{d(x,y)<r} \left(\frac{|u(x) - u(y)|}{r} + \frac{|v(x) - v(y)|}{r} \right)$$

which leads to $\mathrm{Lip}(u+v)(x) \le \mathrm{Lip}\, u(x) + \mathrm{Lip}\, v(x)$. These two are the required properties of a seminorm. □

Definition 8.3.5. *Given any finite ordered set* $\mathbf{u} \subset \mathrm{LIP}(\mathbf{X})$*, thought of as a mapping* $\mathbf{u}\colon X \to \mathbb{R}^n$ *– where* $n = \#\mathbf{u}$ *– and a positive number* δ *define the sets*

1) $S(\mathbf{u}, \delta) = \{x \in X : |\lambda \cdot \mathbf{u}|_x \ge \delta|\lambda| \text{ for all } \lambda \in \mathbb{R}^n\}$ *and*

2) $S(\mathbf{u}) = \{x \in X : |\lambda \cdot \mathbf{u}|_x \neq 0 \text{ for all } \lambda \in \mathbb{R}^n \setminus \{0\}\}$.

Further for any measurable set A *(of positive measure) put*

3) $S_A(\mathbf{u}, \delta) = S(\mathbf{u}, \delta) \cap A$ *and*

4) $S_A(\mathbf{u}) = S(\mathbf{u}) \cap A$

To see that these sets are measurable it is helpful to notice that

Lemma 8.3.6. *for a fixed* $\mathbf{u}$ *and* $x \in X$ *the mapping* $a(x, \cdot) = \lambda \mapsto |\lambda \cdot \mathbf{u}|_x$ *is continuous.*

Proof. If λ and λ' are elements in $\mathbb{R}^n$ where $n = \#\mathbf{u}$, $x \in X$ is fixed and $\mathbf{u}$ is denoted as $\mathbf{u} = (u_1, \ldots, u_n)$ then

$$\Big| |\lambda \cdot \mathbf{u}|_x - |\lambda' \cdot \mathbf{u}|_x \Big| \le |(\lambda - \lambda') \cdot \mathbf{u}|_x \le \sum_{i=1}^{n} |\lambda_i - \lambda_i'| |\mathbf{e}_i \cdot u_i|_x \le C|\lambda - \lambda'|_1$$

□

Lemma 8.3.7. *Both* $S(\mathbf{u}, \delta)$ *and* $S(\mathbf{u})$ *are measurable and further*

$$S(\mathbf{u}) = \bigcup_{i=1}^{\infty} S(\mathbf{u}, 1/i).$$

Proof. Start by showing the measurability of $S(\mathbf{u}, \delta)$ for $\delta > 0$. By Corollary 7.2.3 the map $a_\lambda := a(\cdot, \lambda) = x \mapsto |\lambda \cdot \mathbf{u}|_x - \delta|\lambda|$ is measurable $X \to \mathbb{R}$ and $S(\mathbf{u}, \delta)$ can be written as

$$S(\mathbf{u}, \delta) = \bigcap_{\lambda \in \mathbb{R}^n} a_\lambda^{-1}([0, \infty]).$$

By the continuity of the map $\lambda \mapsto |\lambda \cdot \mathbf{u}|_x$ the union can be taken over Λ which is a countable dense subset of $\mathbb{R}^n$: obviously

$$\bigcap_{\lambda \in \mathbb{R}^n} a_\lambda^{-1}([0, \infty]) \subset \bigcap_{\lambda \in \Lambda} a_\lambda^{-1}([0, \infty])$$

and to see the opposite inclusion suppose $x \in a_\lambda^{-1}([0,\infty])$ for every $\lambda \in \Lambda$ and let $\lambda \in \mathbb{R}^n$ be arbitrary. Then there is a sequence $(\lambda_k) \subset \Lambda$ converging to λ, hence having fixed $x \in X$ lemma 8.3.6 implies the persistence in the limit

$$a(x,\lambda) = \lim_{k\to\infty} a(x,\lambda_k) \in [0,\infty].$$

Therefore $S(\mathbf{u},\delta)$ is measurable.

Now it is evident that

$$S(\mathbf{u}) \supset \bigcup_{i=1}^{\infty} S(\mathbf{u}, 1/i).$$

Again for the opposite inclusion let $x \in S(\mathbf{u})$, i.e. $|\lambda \cdot \mathbf{u}|_x > 0$ for all $\lambda \in \mathbb{R}^n \setminus \{0\}$. The set $\{|\lambda| = 1\}$ is compact, therefore the continuity of $a(x,\cdot)$ implies $\delta := \min_{|\lambda|=1} |\lambda \cdot \mathbf{u}|_x > 0$. Then for any $\lambda \in \mathbb{R}^n$

$$|\lambda \cdot \mathbf{u}|_x \geq \delta|\lambda|$$

and hence $x \in S(\mathbf{u},\delta) \subset S(\mathbf{u},1/i)$ for sufficiently large i. This completes the proof of the lemma. □

Proof of 8.3.1. Let (X,d,μ) be complete and doubling and let K be such that

$$\operatorname{Lip} u(x) \leq K \operatorname{lip} u(x) \text{ for almost every } x \in X$$

for any $u \in \mathrm{LIP}(X)$. The condition

$$\lim_{y\to x} \frac{|u(y) - u(x) - \lambda(x)\cdot(\mathbf{f}(x) - \mathbf{f}(y))|}{d(y,x)} = 0$$

is equivalent to

$$\limsup_{r\to 0} \sup_{d(x,y)<r} \frac{|u(y) - u(x) - \lambda(x)\cdot(\mathbf{f}(x) - \mathbf{f}(y))|}{d(y,x)} = 0$$

since this last limit is just the definition "$\limsup_{y\to x}$". It can therefore be expressed in terms of the seminorm $|\cdot|_x$:

$$|u(\cdot) - \lambda(x)\cdot \mathbf{f}(\cdot)|_x = 0. \tag{8.3.2}$$

Throughout this proof the measurable set A with positive measure will be fixed.

For any $V \subset A$ measurable with positive measure and any ordered set $\mathbf{f}$ for which $\#\mathbf{f} > 0$ and $\mu(S_V(\mathbf{f})) > 0$ (supposing such sets exist) utilize lemma 8.3.7 to find some $\delta > 0$ for which $\mu(S_V(\mathbf{f},\delta)) > 0$. The function $\mathbf{f}$ satisfies

$$\operatorname{Lip}(\lambda\cdot\mathbf{f})(x) \leq K \operatorname{lip}(\lambda\cdot\mathbf{f})(x) \text{ for almost every } x \in X$$

for any $\lambda \in \mathbb{R}^n$ because for each $\lambda \in \mathbb{R}^n$, $\lambda\cdot\mathbf{f} \in \mathrm{LIP}(X)$. Also the set $W := \{x \in V : |\lambda\cdot\mathbf{f}|_x \geq \delta|\lambda| \text{ for all } \lambda \in \mathbb{R}^n\} = S_V(\mathbf{f},\delta)$ has positive measure, namely

$$\mu(W) = \mu(S_V(\mathbf{f},\delta)) > 0.$$

Corollary 8.2.7 then ensures an upper bound for n.

Now define

$$n(A) = \max\{\#\mathbf{f} : \mu(S_A(\mathbf{f})) > 0\}.$$

The maximum is taken over all finite ordered sequences $\mathbf{f} \subset \mathrm{LIP}(X)$ (thought of as functions $\mathbf{f} : X \to \mathbb{R}^{\#\mathbf{f}}$) for which $\mu(S_A(\mathbf{f})) > 0$.

There are two possibilities. One is that $n(A) = 0$ and the other that $n(A) > 0$. Consider the first one.

Set $\mathbf{f} : X \to \{0\}$ to be the constant function in which case the condition (8.3.2) reduces to $|u|_x = 0$. In other words every $u \in \mathrm{LIP}(X)$ should satisfy $\operatorname{Lip} u(x) = 0$ for almost every $x \in V$, V being any positive measured measurable subset of A. Choose $V = A$. Then indeed $\operatorname{Lip} u(x) = 0$ for almost every $x \in A$ since if $\operatorname{Lip} u(x) > 0$ in a set B of positive measure then $\mathbf{g} := \{u\} \subset \mathrm{LIP}(X)$ would constitute an ordered set satisfying $\#\mathbf{g} > 0$ and $\mu(S_A(\mathbf{g})) = \mu(B) > 0$ thus contradicting the assumption that $n(V) = 0$ for all $V \subset A$. It has been established, then, that if $n = 0$ then the constant function $\mathbf{f} : X \to \mathbb{R}^0$ and $V = A$ satisfy the claims of proposition 8.3.1

It can therefore be supposed throughout the rest of this proof that $n(A) > 0$. By the discussion above $n(A)$ is bounded above by a constant depending only on K and the doubling constant of the function. This upper bound for n ensured by Corollary 8.2.7 is in fact the required upper bound in the claim of 8.3.1.

Now fix some ordered set $\mathbf{f} \subset \mathrm{LIP}(X)$ so that $\#\mathbf{f} = n(A) =: n$ and let $\delta > 0$ and W be as above. It remains to prove that with this choice every $u \in \mathrm{LIP}(X)$ admits a "differential" $\mathrm{d}u : W \to \mathbb{R}^n$, i.e. a measurable function that satisfies

$$|u(\cdot) - \mathrm{d}u(x) \cdot \mathbf{f}(\cdot)|_x = 0 \text{ for almost every } x \in W.$$

To accomplish this consider the set

$$E = \{x \in W : |u - \lambda \cdot \mathbf{f}|_x \neq 0 \text{ for all } \lambda \in \mathbb{R}^n\}.$$

The measurability of this can be seen as in the proof of Lemma 8.3.7. Define $\mathbf{f}' = (f_1, \ldots, f_n, u) : X \to \mathbb{R}^{n+1}$. If $x \in E$ and $\lambda' = (\lambda, \lambda_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}$ then

$$|\lambda' \cdot \mathbf{f}'|_x = |\lambda_{n+1} u + \lambda \cdot \mathbf{f}|_x = \begin{cases} \delta|\lambda| & , \lambda_{n+1} = 0 \\ |\lambda_{n+1}||\lambda/\lambda_{n+1} \cdot \mathbf{f} - u|_x & , \text{otherwise,} \end{cases}$$

in particular $x \in S_W(\mathbf{f}') \subset S_A(\mathbf{f}')$. But $\#\mathbf{f}' = n + 1 > n(A)$ and therefore $\mu(S_V(\mathbf{f}')) = 0$.

It follows that the set E has measure zero and therefore that for almost every $x \in W$ there is some $\lambda \in \mathbb{R}^n$ for which $|\lambda \cdot \mathbf{f} - u|_x = 0$. Suppose λ and λ' are two such vectors. Then

$$0 = |\lambda \cdot \mathbf{f} - u|_x + |u - \lambda' \cdot \mathbf{f}|_x \geq |(\lambda - \lambda') \cdot \mathbf{f}|_x \geq \delta|\lambda - \lambda'|$$

because the last inequality holds for all $x \in W$. Consequently for almost every $x \in W$ there exists a ***unique*** $\lambda = \lambda(x) \in \mathbb{R}^n$ so that

$$|\lambda \cdot \mathbf{f} - u|_x = 0. \tag{8.3.3}$$

This defines a unique mapping $\mathrm{d}u : W \to \mathbb{R}^n$ – i.e. $\mathrm{d}u(x) = \lambda(x)$ – up to a set of measure zero – which has the desired property.

The proof will be complete as soon as $\mathrm{d}u : W \to \mathbb{R}^n$ is shown to be measurable. The values of $\mathrm{d}u$ in E do not have an influence on the measurability of $\mathrm{d}u$. Let us define $\mathrm{d}u = 0$ on E.

Let $U \subset \mathbb{R}^n$ be compact and consider the set

$$F = \{x \in W : \text{ there exists some } \lambda \in U \text{ so that } |\lambda \cdot \mathbf{f} - u|_x = 0\}.$$

This is *almost* the pre-image of U in $\mathrm{d}u$, in the sense that

$$\mathrm{d}u^{-1}U = \begin{cases} F & \text{if } 0 \notin U \\ F \cup E & \text{if } 0 \in U. \end{cases}$$

Therefore it suffices to show the measurability of F. Firstly

$$F = \bigcap_{n=1}^{\infty} F_n, \quad F_n = \{x \in W : |\lambda \cdot \mathbf{f} - u|_x < 1/n \text{ for some } \lambda \in U\}.$$

The inclusion "$\subset$" is clear. If, on the other hand, $x \in \bigcap_{n=1}^{\infty} F_n$ then there is a sequence $(\lambda_n) \subset U$ which, by the compactness of U can be assumed to be convergent, so that $|\lambda_n \cdot \mathbf{f} - u|_x < 1/n$ for all n. Denoting by λ the limit of λ_n, the continuity of the mapping $\lambda \mapsto |\lambda \cdot \mathbf{f} - u|_x$ implies $|\lambda \cdot \mathbf{f} - u|_x = 0$.

Furthermore if Λ is some countable dense subset of U and if some x and $\lambda' \in U$ satisfy the relation $|\lambda' \cdot \mathbf{f} - u|_x < 1/n$ then there exists some $\lambda \in \Lambda$ for which $|\lambda \cdot \mathbf{f} - u|_x < 1/n$ holds. Hence F_n can be written as

$$F_n = \bigcup_{\lambda \in \Lambda} F_n^{\lambda}$$

where

$$F_n^{\lambda} = \{x \in W : |\lambda \cdot \mathbf{f} - u|_x < 1/n\} = [\mathrm{Lip}(\lambda \cdot \mathbf{f} - u)]^{-1}[0, 1/n).$$

These sets are cleary measurable (even Borel, since $\mathrm{Lip}\, v$ of a Lipschitz map v is Borel). Consequently

$$F = \bigcap_{n=1}^{\infty} \bigcup_{\lambda \in \Lambda} F_n^{\lambda}$$

is measurable (Borel). □

In the preceding discussion it was established that each measurable set $A \subset X$ with positive measure contains some measurable set W with positive measure for which the conclusions of 8.3.1 hold. In the proof the required set went by the name W instead of V. Nevertheless after assuring the truthfulness of 8.3.1 lemma 8.3.2 provides the actual differential structure with respect to $\mathrm{LIP}(X)$. Concerning the non-degeneracy of the differential structure we have the following result.

Proposition 8.3.8. *Let (X, d, μ) be k-quasiconvex for some some $k > 0$ and let $(X_\lambda, \varphi_\lambda)_{\lambda \in \Lambda}$ be a differential structure in X. Then the differential structure is non-degenerate.*

Proof. Given a point $y_0 \in X$ define

$$u(x) = \inf_{\gamma} \ell(\gamma), \ x \in X.$$

The infimum is taken over all rectifiable curves joining x to y_0. This function is k-Lipschitz. To see this take $x, y \in X$ and suppose $u(x) \geq u(y)$. For any $\varepsilon > 0$ take γ_y to be a curve joining y_0 and y so that $u(y) + \varepsilon \geq \ell(\gamma_y)$ and let γ_{yx} be a curve joining y and x so that $\ell(\gamma_{yx}) \leq kd(x, y)$. Then the composition $\gamma_y\gamma_{yx}$ is a rectifiable curve from y_0 to x. Calculate

$$0 \leq u(x) - u(y) \leq \ell(\gamma_y\gamma_{yx}) - \ell(\gamma_y) + \varepsilon \leq \ell(\gamma_y) + \ell(\gamma_{yx}) - \ell(\gamma_y) + \varepsilon \leq kd(x, y) + \varepsilon.$$

To see that $\operatorname{lip} u(x) \geq 1$ consider the *length metric* on X defined by $l(x, y) = \inf_\gamma \ell(\gamma)$ where the infimum is taken over all rectifiable curves joining x and y. Let $x \in X$ and $0 < r < \min\{l(y_0, x), 1\}$ be arbitrary. Let $\gamma : [0, L] \to X$ be a rectifiable curve joining y_0 and x such that

$$\ell(\gamma) \geq u(x) \geq \ell(\gamma) - r^2.$$

There exists $y_r = \gamma(t)$ so that $\ell(\gamma|_{[0,t]}) = \ell(\gamma) - r$, whence

$$u(x) \geq \ell(\gamma) - r^2 = \ell(\gamma|_{[0,t]}) + r - r^2 \geq u(y_r) + r(1 - r).$$

To estimate the distance of y_r from x observe that $r \geq l(x, y_r) \geq d(x, y_r)$. Thus

$$\liminf_{r\to 0} \sup_{z\in B(x,r)} \frac{|u(x) - u(z)|}{r} \geq \liminf_{r\to 0} \frac{|u(x) - u(y_r)|}{r} \geq 1$$

These two inequalities together imply

$$1 \leq \operatorname{lip} u(x) \leq \operatorname{Lip} u(x) \leq k$$

for all $x \in X$.

Next suppose that $n(\lambda) = 0$ for some coordinate patch X_λ. For such an index $\lambda \in \Lambda$ choose $y_\lambda \in X_\lambda$ and set u_λ as above with $y_0 = y_\lambda$. Since the coordinate function φ_λ has range $\{0\}$ condition (8.0.2) takes the form

$$|u|_x = 0 \text{ for almost every } x \in X_\lambda$$

for each $u \in \mathrm{LIP}(X)$ (similarly as in the proof of 8.3.1). This, however is clearly untrue in the case of u_λ whose seminorm at any point $x \in X_\lambda$ stays safely above 1. Therefore the existence of a function such as the one constructed above guarentees that the differential structure cannot be degenerate. □

In connection with Theorem 7.3.3 Proposition 8.3.8 implies specifically that if (X, d, μ) admits a p-Poincaré inequality for some $p \geq 1$ then this differential structure is non-degenerate.

Corollary 8.3.9. *Suppose (X, d, μ) is path connected and admits a p-Poincaré inequality for some $p \geq 1$ and let $(X_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$ be a differential structure in X. Then the differential structure is non-degenerate.*

As stated, this together with 8.0.4 and 8.3.1 immediately yields

Theorem 8.3.10. *If (X, d, μ) is path connected and supports a p-Póincare inequality for some $p \geq 1$ then it admits a non-degenerate differentiable structure as defined in 8.0.5.*

8.4 Reflexivity of $N^{1,p}(X)$

In this subsection the space of p-integrable vector fields over X, denoted $\mathcal{L}^p(X)$, will be used. In the classical case (where $X = \mathbb{R}^n$) the space $\mathcal{L}^p(X)$ would correspond to $L^p(X;\mathbb{R}^n)$ – the $\mathbb{R}^n$-valued p-integrable (and measurable) mappings $u : X \to \mathbb{R}^n$. With X a metric measure space supporting some Poincaré inequality the situation is more complicated; Here $\mathcal{L}^p(X)$ consists of p-integrable *vector fields* over X. The precise definitions will be given shortly.

The aim of this subsection is to demonstrate that the Newtonian spaces (and consequently the Hajłasz spaces) with exponent $p > 1$, defined over a metric measure space supporting a p-Poincaré condition are reflexive. The strategy for proving this is to construct a linear isomorphism between $N^{1,p}(X)$ and a closed subspace of $L^p(\mu) \times \mathcal{L}^p(X)$ or, rather, a linear map between $N^{1,p}(X)$ and $L^p(\mu) \times \mathcal{L}^p(X)$ that is bounded both above and below (in a similar fashion as was done in section 3). The reflexivity (or even uniform convexity) of $N^{1,p}(X)$ will then follow from that of $L^p(\mu) \times \mathcal{L}^p(X)$.

To define vector fields over X one needs to talk about tangent spaces and -bundles. Let $(X_k, \varphi_k)_{k\in K}$ be a disjoint measurable structure as given by Lemma 8.3.2 (the lemma states that the decomposition can be taken disjoint). Define the disjoint union

$$TX := \bigcup_{k\in K} (X_k \times R^{n(k)})$$

and

$$T_xX := \pi^{-1}(\{x\}),$$

where $\pi : TX \to X$ is the natural projection. These are the *tangent bundle* and *tangent space at the point* x of X with respect to the differentiable structure $(X_k, \varphi_k)_{k\in K}$, given in analogue with the case where X would be some manifold. Since the union of X_k's covers only almost all of X the tangent space T_xX exists for only almost every $x \in X$. Suppose $k \in K$ and consider the function $\varphi_k(x) = (\varphi_k^1(x), \ldots, \varphi_k^{n(k)}(x))$. Since

$$\varphi_k^i(x) - \varphi_k^i(y) = e_i \cdot (\varphi_k(x) - \varphi_k(y))$$

for every $y \in X_k$ it follows that $\mathrm{d}_k\varphi_k^i(x) = e_i$ for almost every $x \in X_k$, for each $1 \le i \le n(k)$. For such x then take $\left\{\mathrm{d}_k\varphi_k^i(x)\right\}_{i=1}^{n(k)}$ as the basis for T_xX (which, recall, is isomorphic with $\mathbb{R}^{n(k)}$) and define a norm in T_xX by

$$||\lambda||_x = \left|\sum_{i=1}^{n(k)} \lambda^i \varphi_k^i\right|_x = |\lambda \cdot \varphi_k|_x$$

That this is a norm follows from the fact that $|\lambda \cdot \varphi_k|_x > 0$ for $\lambda \neq 0$ almost everywhere in X_k.

With these notations a vector field ω on X is a map $\omega : X \to TX$ such that $\omega(x) \in T_xX$ almost everywhere.

In particular at almost every point $x \in X_k$ then the differential $\mathrm{d}_k u$ of a Lipschitz function u has a value in T_xX;

$$\mathrm{d}_k u(x) = \sum_{i=1}^{n(k)} \mathrm{d}_k^i u(x) \mathrm{d}_k \varphi_k^i(x) \in T_xX$$

and thus d_k is a mapping from X_k to TX.

A *measurable* vector field ω is a vector field whose components in each patch X_k are measurable in the usual sense. That is, if ω is restricted to any X_k and written as $\omega(x) = \sum_{i=1}^{n(k)} \omega^i(x)\mathrm{d}_k\varphi_k^i(x)$ then each ω^i is a measurable mapping $X \to \mathbb{R}$.

Define $\mathcal{L}(X)$ to be the set of measurable vector fields ω on X. For these the map $x \mapsto ||\omega(x)||_x$ is measurable. $\mathcal{L}^p(X)$ then consists of those elements $\omega \in \mathcal{L}(X)$ for which the mapping $x \mapsto ||\omega(x)||_x$ is p-integrable. For this integral the following shorthand notation will be used

$$\left(\int_X \|\omega(x)\|_x^p \mathrm{d}\mu(x)\right)^{1/p} =: \|\omega\|_{\mathcal{L}^p}.$$

This defines, in the usual way, a norm in $\mathcal{L}^p(X)$ which makes it a Banach space.

Lemma 8.4.1. *Let $p \in (1, \infty)$. Then the space $\mathcal{L}^p(X)$ is a Banach space. Furthermore it is reflexive.*

Proof. Start by writing the norm of an element $\omega \in \mathcal{L}^p(X)$ as

$$\|\omega\|_{\mathcal{L}^p(X)}^p = \sum_{k\in K} \int_{X_k} \|\omega(x)\|_x^p \mathrm{d}\mu(x) = \sum_{k\in K} \|\omega|_{X_k}\|_{\mathcal{L}^p(X_k)}. \tag{8.4.1}$$

Conversely every sequence $(\omega_k)_{k\in K}$ for which $\omega_k \in \mathcal{L}^p(X_k)$ and the right-hand side of (8.4.1) is finite determines an element $\omega \in \mathcal{L}^p(X)$ in the obvious way, such that (8.4.1) holds. Hence

$$\mathcal{L}^p(X) = \bigoplus_{\ell^p(K)} \mathcal{L}^p(X_k)$$

with equal norms. Now both claims will follow from Theorem 2.2 if the spaces $Y_k := \mathcal{L}^p(X_k)$ can be shown to be Banach and reflexive.

To this purpose let

$$L_k = \max\{\mathrm{LIP}(\varphi_k^i) : 1 \le i \le n(k)\}$$

and let δ_k be as in Definition 8.0.5, for $k \in K$ fixed. Regarding $\omega = (\omega_1, \dots, \omega_{n(k)})$ as a vector in $\mathbb{R}^{n(k)}$ the estimate

$$\delta_k|\omega(x)| \le \|\omega(x)\|_x \le \sum_{i=1}^{n(k)} |\omega^i(x)|\|\varphi_k^i(x)\|_x \le n(k)L_k|\omega(x)|$$

holds for almost every $x \in X_k$. Here $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^{n(k)}$. As a consequence of this the norm $\|\cdot\|_{\mathcal{L}^p(X_k)}$ is equivalent to $\|\cdot\|_{L^p(X_k;\mathbb{R}^{n(k)})}$. In particular

$$\mathcal{L}^p(X_k) \approx L^p(X_k; \mathbb{R}^{n(k)})$$

as Banach spaces. The reflexivity of each Y_k follows. □

The next proposition gives some more information of the differential, the non-trivial part of the isomorphism that is the purpose of this discussion. The first natural domain of definition for this is the space $\mathrm{LIP}(X)$. The next lemma glues together the pieces given above to establish this.

Proposition 8.4.2. *Let $(X_k, \varphi_k)_{k\in K}$ be a differential structure over (X, d, μ) as above. Then there is an operator $\mathrm{d} : \mathrm{LIP}(X) \to \mathcal{L}(X)$ satisfying the following conditions.*

a) For each $u \in \mathrm{LIP}(X)$ and $k \in K$ one has $|u - \mathrm{d}u(x) \cdot \varphi_k|_x = 0$ for almost every $x \in X_k$.

b) $\mathrm{d}(u + v) = \mathrm{d}u + \mathrm{d}v$ and $\mathrm{d}(au) = a\mathrm{d}u$ almost everywhere for any $u, v \in \mathrm{LIP}(X)$ and $a \in \mathbb{R}$.

c) For each $u \in \mathrm{LIP}(X)$ the identity $\|\mathrm{d}(x)\|_x = \mathrm{Lip}\, u(x)$ holds almost everywhere.

d) If $u \in \mathrm{LIP}(X)$ and $f : \mathbb{R} \to \mathbb{R}$ is a differentiable mapping, then $\mathrm{d}(f \circ u) = f'(u)\mathrm{d}u$ almost everywhere.

Proof. Let $u \in \mathrm{LIP}(X)$. Define $\mathrm{d}u : X \to TX$ almost everywhere by

$$\mathrm{d}u(x) := \mathrm{d}_k u(x)$$

where $k \in K$ is the unique index for which $x \in X_k$. This indeed gives a(n almost everywhere defined) function $\mathrm{d}u : X \to TX$. Hence d can be thought of as an operator from $\mathrm{LIP}(X)$ to the set of (measurable) vector fields over X.

Condition a) is automatic because for any $u \in \mathrm{LIP}(X)$ almost every $x \in X$ belongs to X_k for a unique $k \in K$ $\mathrm{d}u(x) = \mathrm{d}_k u(x)$ and thus (8.0.2) holds almost everywhere in X_k.

To prove b) take $u, v \in \mathrm{LIP}(X)$ and $a \in \mathbb{R}$ and suppose x is such that condition a) holds for both u and v. Let k be such that $x \in X_k$. In the proof of 8.3.1 it was seen that once the function φ_k is fixed this condition determines $\mathrm{d}_k u(x)$ uniquely. Bearing this in mind the computation

$$\begin{aligned}&|u + v - (\mathrm{d}u(x) + \mathrm{d}v(x)) \cdot \varphi_k|_x = |u + v - (\mathrm{d}_k u(x) + \mathrm{d}_k v(x)) \cdot \varphi_k|_x \\ &\le |u - \mathrm{d}_k u(x) \cdot \varphi_k|_x + |v - \mathrm{d}_k v(x) \cdot \varphi_k|_x = 0\end{aligned}$$

implies that $\mathrm{d}(u + v)(x) = \mathrm{d}u(x) + \mathrm{d}v(x)$. Similarly

$$|au - a\mathrm{d}_k u(x) \cdot \varphi_k|_x = |a||u - \mathrm{d}_k u(x) \cdot \varphi_k|_x = 0,$$

hence $\mathrm{d}(au) = a\mathrm{d}u$.

For almost every $x \in X$ one has

$$\|\mathrm{d}u(x)\|_x = \left|\sum_{i=1}^{n(k)} \mathrm{d}^i u(x)\varphi_k^i\right|_x = |\mathrm{d}u(x) \cdot \varphi_k|_x$$

On the other hand condition a) implies $|u|_x = |\mathrm{d}u(x) \cdot \varphi_k|_x$ almost everywhere since on every point where a) holds it leads to

$$||u|_x - |\mathrm{d}u(x) \cdot \varphi_k|_x| \le |u - \mathrm{d}u(x) \cdot \varphi_k|_x = 0$$

For d) let $u \in \mathrm{LIP}(X)$ and let x be a point where a) holds for u. Then $f \circ u(y) = f \circ u(x) + f' \circ u(x)[u(y) - u(x)] + o(d(x,y))$, yielding

$$f \circ u(y) - f' \circ u(x) \mathrm{d}u(x) \cdot \varphi_k(y) =$$
$$f \circ u(x) - u(x) f' \circ u(x) + o(d(x,y)) + f' \circ u(x)[u(y) - \mathrm{d}u(x) \cdot \varphi_k(y)].$$

From this expression it is clearly seen that

$$|f \circ u - f' \circ u(x) \mathrm{d}u(x) \cdot \varphi_k|_x = f' \circ u(x) |u - \mathrm{d}u(x) \cdot \varphi_k|_x = 0$$

□

Hence one can consider an "exterior derivative" d on $\mathrm{LIP}(X)$. The ultimate goal in the ongoing discussion is to extend d to an operator between Banach spaces. For this reason it is convenient to present a new equivalent norm on $N^{1,p}(X)$. This is first done in a subset of $N^{1,p}(X)$. The whole space is then obtained by completing this subset. Consequently another characterization of $N^{1,p}(X)$ is attained and this will be used to construct the isomorphism discussed earlier.

Lemma 8.4.3. *Consider the set*

$$N_L^{1,p}(X) = \{u \in \mathrm{LIP}(X) \cap L^p(\mu) : \mathrm{Lip}\, u \in L^p(\mu)\} =$$
$$\{u \in \mathrm{LIP}(X) \cap L^p(\mu) : \mathrm{d}u \in \mathcal{L}^p(X)\}.$$

On it define a norm

$$\|u\|_{1,p} := \|u\|_p + \|\mathrm{Lip}\, u\|_p = \|u\|_{L^p} + \|\mathrm{d}u\|_{\mathcal{L}^p}.$$

Then

1) $\|\cdot\|_{1,p}$ *and* $\|\cdot\|_{N^{1,p}(X)}$ *are equivalent on* $N_L^{1,p}(X)$, *that is for some constant* $C > 0$

$$1/C \|u\|_{1,p} \leq \|u\|_{N^{1,p}(X)} \leq C \|u\|_{1,p}$$

holds for every $u \in N_L^{1,p}(X)$, *and further*

2) $N^{1,p}(X)$ *is obtained as the completion with respect to the norm* $\|\cdot\|_{1,p}$.

In particular the norms $\|\cdot\|_{N^{1,p}(X)}$ *and* $\|\cdot\|_{1,p}$ *are equivalent and* $u \in N^{1,p}(X)$ *if and only if there is a sequence* $u_n \in N_L^{1,p}(X)$ *and* $\omega \in \mathcal{L}^p(X)$ *so that* $||u - u_n||_p \to 0$ *and* $||\mathrm{d}u_n - \omega||_{\mathcal{L}^p} \to 0$ *as* $n \to \infty$.

Proof. By Proposition 7.2.6 for any $v \in N_L^{1,p}(X)$ and its upper gradient g the inequality

$$1/C\, \mathrm{Lip}\, v(x) \leq \limsup_{r \to 0} \frac{1}{r} \fint_{B(x,r)} |v - v_{B(x,r)}| \leq g(x)$$

holds almost everywhere. Hence in particular $\|v\|_{N^{1,p}(X)} \geq \|v\|_p + C^{-1} \|\mathrm{Lip}\, v\|_p$. On the other hand the reverse inequality $\|v\|_{N^{1,p}(X)} \leq \|v\|_p + \|\mathrm{Lip}\, v\|_p$ must hold since $\mathrm{Lip}\, v$ itself is an upper gradient of v. Consequently the norms $\|\cdot\|_{N^{1,p}(X)}$ and $\|\cdot\|_{1,p}$ are equivalent on $N_L^{1,p}(X)$. (In particular $N_L^{1,p}(X)$ is a subset of $N^{1,p}(X)$.)

Denote by H the completion of $N_L^{1,p}(X)$ with respect to $\|\cdot\|_{1,p}$. For any $u \in H$ there exists a sequence $u_n \in N_L^{1,p}(X)$ so that $\|u - u_n\|_{1,p} \to 0$ as $n \to \infty$. The sequence u_n is a Cauchy sequence in $\|\cdot\|_{1,p}$ and by the previous discussion also for $\|\cdot\|_{N^{1,p}(X)}$. Therefore u_n has a limit $\tilde{u} \in N^{1,p}(X)$. By passing to a subsequence it can be assumed that $u_n \to u$ and $u_n \to \tilde{u}$ pointwise almost everywhere. Therefore $u = \tilde{u}$ almost everywhere. This shows that $H = N^{1,p}(X)$ as sets. Further

$$\|u\|_{1,p} = \lim_{n\to\infty} \|u_n\|_{1,p} \leq C \lim_{n\to\infty} \|u_n\|_{N^{1,p}(X)} = C\|u\|_{N^{1,p}(X)}$$

for a sequence $(u_n) \subset N_L^{1,p}(X)$ converging to u. Note that convergence in $\|\cdot\|_{N^{1,p}(X)}$ implies convergence in $\|\cdot\|_{1,p}$ and vice versa. A similar argument can be applied to the other inequality and consequently the norms are seen to be equivalent.

The rest of the claim is a straightforward consequence of 1) and 2) and the fact that $\mathcal{L}^p(X)$ is complete. □

By definition of the space $N_L^{1,p}(X)$ the differential d can be considered a linear map from $N_L^{1,p}(X)$ to $\mathcal{L}^p(X)$. Self-evidently it is also bounded so that it can be extended to a bounded linear operator $\mathrm{d} : N^{1,p}(X) \to \mathcal{L}^p(X)$.

The representation of the differential d as above is the main step in a construction of a bi-Lipschitz linear map between $N^{1,p}(X)$ and $L^p(\mu) \times \mathcal{L}^p(X)$. Note that the space $L^p(\mu) \times \mathcal{L}^p(X)$ can naturally be thought of as a normed vector space endowed with the norm

$$\|(u,\omega)\| := \|(u,\omega)\|_{L^p(\mu)\times\mathcal{L}^p(X)} := \|u\|_p + \|\omega\|_{\mathcal{L}^p}.$$

As a finite cartesian product of two Banach spaces $L^p(\mu) \times \mathcal{L}^p(X)$ is also Banach. The reflexivity of $L^p(\mu) \times \mathcal{L}^p(X)$ follows from that of $L^p(\mu)$ and $\mathcal{L}^p(X)$.

Definition 8.4.4. *Define the linear map $L : N^{1,p}(X) \to L^p(\mu) \times \mathcal{L}^p(X)$ by $L(u) = (u, \mathrm{d}u)$.*

Definition 8.4.4 now implies the desired result by a very elementary argument

Corollary 8.4.5. *The operator L is an isometry when $N^{1,p}(X)$ is endowed with $\|\cdot\|_{1,p}$. Consequently the space $N^{1,p}(X)$ is reflexive.*

Proof.

$$\|Lu\|_{L^p(\mu)\times\mathcal{L}^p(X)} = \|u\|_p + \|\mathrm{d}u\|_p = \|u\|_{1,p}.$$

Hence $N^{1,p}(X)$ is isomorphic with a closed subspace of $L^p(\mu) \times \mathcal{L}^p(X)$ which implies the reflexivity of $N^{1,p}(X)$. □

Bearing in mind that the spaces $N^{1,p}(X)$ and $M^{1,p}(X)$ are isomorphic when $p > 1$ the following main result is obtained.

Theorem 8.4.6. *If (X, d, μ) is a complete doubling space and supports a p-Poincaré inequality for some $p > 1$ then the spaces $N^{1,p}(X)$ and $M^{1,p}(X)$ defined over X are isomorphic (to each other) and reflexive.*

8.5 Remarks

The construction of the differentiable structure could, with not much additional work, have been carried out for arbitrary families of Lipschitz functions under the same hypotheses. Also the construction of the "manifold" structure of X at almost every point does not require the disjointness of the charts X_k but this is rather a convenient assumption which allows for a less messy treatment of the subject (and is sufficient for the purposes of proving the ultimate result). The work done here was aimed at proving the *reflexivity* of the spaces in question. In fact the space $\mathcal{L}^p(X)$ (and consequently $N^{1,p}(X)$) can be equipped with an equivalent norm that is uniformly convex. To accomplish this some more work is needed.

Theorem 8.5.1. *The space $\mathcal{L}^p(X)$ is uniformly convex.*

To prove this the following lemma, known as John's ellipsoid theorem, will be used. The claim is of elementary nature but it takes some doing to prove it. A proof can be found for example in [18], see also [11] and [9].

Lemma 8.5.2. *Let $K \subset \mathbb{R}^m$ be a symmetric, compact convex set with nonempty interior (that is, a symmetric* body*). Then there exists an ellipsoid $D \subset K$, symmetric about the origin so that $K \subset \sqrt{m}D$.*

Proof of lemma 8.5.1. The objective is to construct, for almost every $x \in X$ equivalent norms given by an inner product in T_xX so that the equivalence constants depend only on the dimension of T_xX. These will lead to an equivalent norm on $\mathcal{L}^p(X)$ and a version of the Hölder inequality that will imply the reflexivity.

Fix some $x \in X$ for which the normed space $(T_xX, \|\cdot\|_x)$ can be constructed. Let m be the dimension of T_xX – that is, $m = n(k)$ where k is such that $x \in X_k$. Every ellipsoid in $\mathbb{R}^m$ that is symmetric about the origin gives rise to an inner product in $\mathbb{R}^m$. In particular if K is the unit ball with respect to the norm $\|\cdot\|_x$ then K is a symmetric body and the ellipsoid D given by John's theorem determines an inner product, denoted $\langle\cdot,\cdot\rangle_x$. The inclusions $D \subset K \subset \sqrt{m}D$ transform to

$$[a]_x \leq \|a\|_x \leq \sqrt{m}[a]_x \text{ for all } a \in T_xX$$

in the context of norms. Here $[\cdot]_x$ is the norm determined by the inner product. Since $m = n(k)$ has a uniform (in k) upperbound there exists a constant depending only on the dimension of the differential structure such that $1/C\|a\|_x \leq [a]_x \leq C\|a\|_x$ for almost every $x \in X$ and every $a \in T_xX$. In particular $\|\cdot\|_{\mathcal{L}^p}$ is equivalent with

$$\|\cdot\| := \omega \mapsto \left(\int_X [\omega(x)]_x^p \mathrm{d}\mu(x)\right)^{1/p},$$

provided that the mapping $x \mapsto [\omega]_x$ is measurable for fixed ω. For the moment this will merely be assumed and this question will be visited in the sequel.

The following lemma will be used to prove that $\|\cdot\|$ is uniformly convex.

Lemma 8.5.3. *For any two measurable vector fields ω and σ and for almost every $x \in X$ the following inequalities hold.*

$$[\omega(x)+\sigma(x)]_x^p + [\omega(x)-\sigma(x)]_x^p \leq 2^{p-1}([\omega(x)]_x^p + [\sigma(x)]_x^p) \tag{8.5.1}$$

if $p \geq 2$, *and*

$$([\omega(x) + \sigma(x)]_x^q + [\omega(x) - \sigma(x)]_x^q)^{p-1} \leq 2^{p-1}([\omega(x)]_x^p + [\sigma(x)]_x^p) \qquad (8.5.2)$$

if $1 < p \leq 2$. *Here* $q = \dfrac{p}{p-1}$ *is the Hölder conjugate exponent of* p.

Proof. Note that in the case $p = 2$ equality holds in (8.5.1) by the usual parallelogram law of inner products. Let $p \geq 2$ and estimate

$$\begin{aligned}
&([\omega(x) + \sigma(x)]_x^2)^{p/2} + ([\omega(x) - \sigma(x)]_x^2)^{p/2} \\
&\leq ([\omega(x) + \sigma(x)]_x^2 + [\omega(x) - \sigma(x)]_x^2)^{p/2} = (2[\omega(x)]_x^2 + 2[\sigma(x)]_x^2)^{p/2} \\
&= 2^p \left(\frac{[\omega(x)]_x^2 + [\sigma(x)]_x^2}{2} \right)^{p/2}.
\end{aligned}$$

Using the convexity of the mapping $t \mapsto t^{p/2}$ yields the final result:

$$[\omega(x)+\sigma(x)]_x^p + [\omega(x)-\sigma(x)]_x^p \leq 2^p \left(\frac{[\omega(x)]_x^2 + [\sigma(x)]_x^2}{2} \right)^{p/2} \leq 2^p \frac{[\omega(x)]_x^p + [\sigma(x)]_x^p}{2}.$$

For the case $1 < p \leq 2$ note that $q \geq 2$. It is good to keep in mind the identities

$$\begin{aligned}
&p = q(p-1) \\
&q = p(q-1) \\
&(p-1)(q-1) = 1
\end{aligned}$$

which will be used without further mention. Start with the inequality

$$(s+t)^q + (s-t)^q \leq 2(s^p + t^p)^{q-1},$$

valid for $0 \leq s \leq t$ and $1 < p \leq 2$. The proof for this can be found in ([3], Th. 2). (Notice that this is a special case of (8.5.2).) To prove (8.5.2) choose an orthonormal basis for $\mathbb{R}^n$ with respect to the inner-product norm $[\cdot]_x$ and write the norm (squared) of any vector field as a sum of the squares of its components. Raising both sides of (8.5.2) to the power $q-1$ the claim then takes the equivalent form

$$\begin{aligned}
&\left(\sum_{k=1}^n |\omega^i + \sigma^i|^2 \right)^{q/2} + \left(\sum_{k=1}^n |\omega^i - \sigma^i|^2 \right)^{q/2} \\
&\leq 2 \left(\left(\sum_{k=1}^n |\omega^i|^2 \right)^{p/2} + \left(\sum_{k=1}^n |\sigma^i|^2 \right)^{p/2} \right)^{q-1}.
\end{aligned}$$

Write the left side of this as

$$\left(\sum_{k=1}^n (|\omega^i + \sigma^i|^q)^{2/q} \right)^{q/2} + \left(\sum_{k=1}^n (|\omega^i - \sigma^i|^q)^{2/q} \right)^{q/2}$$

and use the reverse Minkowski inequality ([15], pp. 146, Thm 198) for $2/q \leq 1$ to estimate this by

$$\left(\sum_{k=1}^{n}(|\omega^i+\sigma^i|^q+|\omega^i-\sigma^i|^q)^{2/q}\right)^{q/2} \leq$$
$$\left(\sum_{k=1}^{n}2^{2/q}(|\omega^i|^p+|\sigma^i|^p)^{(q-1)2/q}\right)^{q/2} = 2\left(\sum_{k=1}^{n}(|\omega^i|^p+|\sigma^i|^p)^{2/p}\right)^{(q-1)p/2}.$$

Since $2/p \geq 1$, application of the normal Minkowski inequality yields an upper estimate of

$$2\left(\left(\sum_{k=1}^{n}(|\omega^i|^p)^{2/p}\right)^{p/2}+\left(\sum_{k=1}^{n}(|\sigma^i|^p)^{2/p}\right)^{p/2}\right)^{(q-1)}$$
$$=2\left(\left(\sum_{k=1}^{n}|\omega^i|^2\right)^{p/2}+\left(\sum_{k=1}^{n}|\sigma^i|^2\right)^{p/2}\right)^{q-1}.$$

This finishes the proof. □

With the aid of the inequality the above Lemma the uniform convexity of $\|\cdot\|$ can be proven. For the case $p \geq 2$ it immediately implies

$$\|\omega+\sigma\|^p+\|\omega-\sigma\|^p \leq 2^{p-1}(\|\omega\|^p+\|\sigma\|^p)$$

for $\omega, \sigma \in \mathcal{L}^p(X)$. Now if $\|\omega\| = \|\sigma\| = 1$ and $\|\omega-\sigma\| = \varepsilon$ for a given $0 < \varepsilon \leq 2$ then

$$\|\omega+\sigma\|^p+\|\omega-\sigma\|^p \leq 2^p$$

which then implies

$$\frac{\|\omega+\sigma\|}{2} \leq \left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}.$$

Consequently

$$\delta(\varepsilon) = \inf\left\{1-\frac{\|\omega+\sigma\|}{2} : \|\omega-\sigma\| = \varepsilon, \|\omega\| = \|\sigma\| = 1\right\} \geq 1-\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}.$$

To address the case $1 < p \leq 2$ one more Corollary is needed.

Corollary 8.5.4. *For $1 < p \leq 2$ one has the inequality*

$$\|\omega+\sigma\|^q+\|\omega-\sigma\|^q \leq 2(\|\omega\|^p+\|\sigma\|^p)^{q-1}$$

for every $\omega, \sigma \in \mathcal{L}^p(X)$. Here $q = \dfrac{p}{p-1}$ is the Hölder conjugate exponent of p.

Proof. Note that $q/p = q-1$ and $p/q = p-1 \leq 1$. Use a reverse Minkowski inequality to obtain

$$\|\omega+\sigma\|^q + \|\omega-\sigma\|^q = \left(\int_X ([\omega+\sigma]^q)^{p/q} \mathrm{d}\mu)\right)^{q/p} + \left(\int_X ([\omega-\sigma]^q)^{p/q} \mathrm{d}\mu\right)^{q/p}$$

$$\leq \left(\int_X ([\omega+\sigma]^q + [\omega-\sigma]^q)^{p/q} \mathrm{d}\mu\right)^{q/p}.$$

Now apply (8.5.2) to the integrand to get the desired result

$$\leq \left(\int_X 2^{p-1}([\omega]^p + [\sigma]^p)\mathrm{d}\mu\right)^{q-1} = 2(\|\omega\|^p + \|\sigma\|^p)^{q-1}.$$

□

The above Corollary implies in a similar manner to the case $p \geq 2$, that

$$\delta(\varepsilon) \geq 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^q\right)^{1/q}.$$

□

It still remains to show that if ω is measurable for some fixed then the mapping $x \mapsto [\omega(x)]_x$ obtained from the new norm is measurable. Loosely speaking this means that the choice of the ellipsoid, or the innerproduct norm can be made in a somehow measurable manner. The trouble is specifying what exactly is meant by "somehow measurable manner". To address the measurability question another version of John's lemma 8.5.2 will be of use. The proof and more related discussion of this can be found in [27] and [19].

Lemma 8.5.5. *Let $\|\cdot\|$ be a norm in $\mathbb{R}^m$, $K \subset \mathbb{R}^m$ its unit ball and $[\cdot]$ the dual norm of $\|\cdot\|_*$, a norm in $(\mathbb{R}^m)^*$ given by*

$$\|u\|_* = \left(⨍_K u(x)^2 \mathrm{d}x\right)^{1/2}.$$

(Both $\|\cdot\|_*$ and $[\cdot]$ are given by an inner product.) Then there is a constant $c = c(m)$ depending only on the dimension m so that

$$\frac{1}{c}\|a\| \leq [a] \leq c\|a\| \text{ for every } a \in \mathbb{R}^m.$$

Returning to the measurability question consider the above norm $[\cdot]_x$ for every $x \in X_k$, ($k \in K$ fixed) for which $(\mathbb{R}^{n(k)}, \|\cdot\|_x)$ can be formed, and let $m = n(k)$. Given $\omega \in \mathcal{L}(X)$ the purpose is to show that $x \mapsto [\omega(x)]_x : X_k \to \mathbb{R}$ is measurable, if $[\cdot]_x$ is chosen as in Lemma 8.5.5. Let $(a_j)_{j=1}^\infty$ be an enumeration of $\mathbb{Q}^m$. For each $n \in \mathbb{N}$ define the mappings

$$p_n^1 : X_k \to \mathbb{R}^n,\ p_n^1(x) = (\|a_1\|_x, \ldots, \|a_n\|_x) \text{ and}$$
$$p_n^2 : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^m),\ p_n^2(y_1, \ldots, y_n) = \operatorname{conv}\{a_i : y_i < 1, i = 1, \ldots, n\}$$

for almost every $x \in X_k$. Clearly p_n^1 can be modified to be Borel (define it to be, say, zero on a null set). The function p_n^2 chooses those points $a_j \in \mathbb{Q}^m$ for which the corresponding jth coordinate is less than one and forms the convex hull of those points that are chosen. Now for sufficiently large n the set $p_n^2 \circ p_n^1(x)$ is convex with nonempty interior: the convexity is of course obvious since $p_n^2(y)$ is convex for every $y \in \mathbb{R}^n$. Let $L \in \mathbb{Q}$ be larger than the maximum of the Lipschitz constants of the functions $e_i \cdot \varphi_k$, $i = 1, \ldots, m$ whence $\|e_i\|_x < L$. This constant does not depend on x. Since $\|e_i/L\|_x < 1$ and furthermore $e_i/L, -e_i/L \in \mathbb{Q}^m$ for every $i = 1, \ldots, m$ there exists some j_0 so that $p_n^2 \circ p_n^1(x)$ is the convex hull of a set of points containing the e_i/L's and the $-e_i/L$'s whenever $n \geq j_0$. Hence $p_n^2 \circ p_n^1(x)$ is a convex set with nonempty interior for all $n \geq j_0$.

Denote by $\mathcal{B}(m)$ the set of all convex subsets of $\mathbb{R}^m$ with nonempty interior. If $B \in \mathcal{B}(m)$ then

$$\langle u, v\rangle_B := \fint_B u(x)v(x)\mathrm{d}x$$

defines an inner produt in $(\mathbb{R}^m)^*$. The dual inner product of this can be identified with a positive definite symmetric matrix $p^3(B)$. This construction also defines the mapping $p^3 : \mathcal{B}(m) \to \mathrm{sym}_+(m)$ where $\mathrm{sym}_+(m)$ denotes the set of positive definite symmetric m by m -matrices. In particular $p^3 \circ p_n^2 \circ p_n^1(x)$ is well defined for sufficiently large n ($n \geq j_0$). For all such n consider the image $p_n^2(\mathbb{R}^n)$. This consists of sets $\mathrm{conv}\{a_{i_1}, \ldots, a_{i_k}\}$ where $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ and there are only finitely many of these. Hence $p_n^2(\mathbb{R}^n)$ is a finite set and consequently $p^3 \circ p_n^2(\mathbb{R}^n)$ is a finite set. For any $M \subset \mathrm{sym}_+(m)$ the set $(p^3 \circ p_n^2)^{-1}(M)$ is then Borel: obviously $(p^3)^{-1}(M) \cap p_n^2(\mathbb{R}^n)$ is finite, say $\{B_1, \ldots, B_r\}$. If $B_i = \mathrm{conv}\{a_{i_1}, \ldots, a_{i_k}\} \in (p^3)^{-1}(M) \cap p_n^2(\mathbb{R}^n)$ then $(p_n^2)^{-1}(B)$ is a cartesian product of intervals that are either $(-\infty, 1)$, $[1, \infty)$ or $(-\infty, \infty)$ (if $j \neq i_l$ for some l then the interval corresponding to the jth coordinate is $[1, \infty)$ whereas for $j = i_l$ for some l the interval corresponding to the jth axis is either $(-\infty, 1)$ or,if it so happens that a_{i_l} lies in the line-segment connecting two a_j's with j appearing in the definition of B_i, then it doesn't matter whether or not this element is chosen in the definition of p_n^2 and therefore the interval corresponding to that is $(-\infty, \infty)$). The preimage can now be expressed as

$$(p^3 \circ p_n^2)^{-1}(M) = \bigcup_{i=1}^{r} (p_n^2)^{-1}(B_r)$$

which is clearly Borel.

Hence $p^3 \circ p_n^2$ and consequently $p_n := p^3 \circ p_n^2 \circ p_n^1$ is a Borel function. To accomplish the purpose of all this – which was to prove the measurability of $x \mapsto [\omega]_x$ – it suffices to prove that $p_n(x) \to p(x)$ almost everywhere where $p : X_k \to \mathrm{sym}_+(m)$ attaches to almost every $x \in X_k$ the inner product norm $[\cdot]_x$ in $\mathbb{R}^m$ described in 8.5.5. The sufficiency can be seen as follows: as a pointwise limit of Borel functions p is measurable. Therefore its coordinate projections $p_{ij} : X_k \to \mathbb{R}$, $i, j = 1, \ldots, m$ are measurable. In particular, if $\langle \cdot, \cdot \rangle_x$ denotes the inner product inducing $[\cdot]_x$, the coordinate projections satisfy

$$p_{ij}(x) = \langle e_i(x), e_j(x)\rangle_x$$

where $e_i(x) = \mathrm{d}\varphi_k^i(x)$, $i = 1, \ldots, m$ is the standard basis of $\mathbb{R}^m$. Therefore $[\omega(x)]_x$ can be expressed as

$$[\omega(x)]_x = \sqrt{\sum_{i,j} p_{ij}(x)\omega^i(x)\omega^j(x)}$$

which shows the desired measurability since ω^i is measurable for each $i = 1, \ldots, m$.

Now to prove the pointwise convergence consider, for almost every $x \in X_k$ the open unit ball $K \subset \mathbb{R}^m$ of $\| \cdot \|_x$ and let $K_n = p_n^2 \circ p_n^1(x)$. Since K_n is by definition the smallest convex set containing the points a_i, $i = 1, \ldots, m$ for which $\|a_i\|_x < 1$ it follows that $K_n \subset K$. In addition the sequence K_n is increasing in the sense that $K_n \subset K_{n+1}$ for all n. Now the density of $(a_j)_{j \in \mathbb{N}}$ in $\mathbb{R}^m$ implies that

$$K = \bigcup_{n=1}^{\infty} K_n.$$

To see this take $y \in K$ which then lies in the line segment between some $a_i, a_j \in \mathbb{Q}^m$, which in turn lie in the halfline $\{ty : t \geq 0\}$, hence $y \in K_n$ for $n \geq \max\{i, j\}$. Then for all $u, v \in (\mathbb{R}^m)^*$

$$\fint_{K_n} u(y)v(y)\mathrm{d}y \overset{n \to \infty}{\longrightarrow} \fint_K u(y)v(y)\mathrm{d}y.$$

From the definition of the dual norm it is seen that likewise $p_n(x)(y, z) \to p(x)(y, z)$ for all $y, z \in \mathbb{R}^m$, that is $p_n(x) \to p(x)$ as $n \to \infty$.

This completes the proof of Theorem 8.5.1 by addressing the measurability of $x \mapsto [\omega(x)]_x$. In fact in the proof of 8.5.1 $[\cdot]_x$ should mean the inner product norm given by lemma 8.5.5. The quantitative results of course remain the same. Notice, however, that the e^i's and ω^i's appearing in the above discussion are different from those appearing in the proof 8.5.1.

It is now an immediate consequence of ([3], Th. 1) that if $N^{1,p}(X) = M^{1,p}(X) =: H^{1,p}(X)$ is equipped with the norm

$$\|u\|_{H^{1,p}}^p = \int_X |u|^p \mathrm{d}\mu + \int_X [\mathrm{d}u]^p \mathrm{d}\mu$$

then $H^{1,p}(X)$ is uniformly convex.

In short, complete doubling metric spaces supporting some Poincaré inequality enjoy a surprisingly large amount of the first order smoothness usually associated to Euclidean spaces (and their smooth subsets). In this thesis this is most strongly displayed by the nice behaviour of the associated various Sobolev type spaces.

However for some results, supporting a Poincaré inequality is a needlessly strong assumption. The construction of the differential structure, as can readily be noted, only requires that the metric space in question satisfies the conclusion of Theorem 8.0.4, for a uniform constant K. (Although this assumption does not guarantee the quasiconvexity of the space, and hence the non-degeneracy of the differential structure.) The reflexivity and the uniform boundedness, on the other hand, only requires that the space admits a differentiable structure (an a priori weaker condition than even the inequality of Theorem 8.0.4). [9]

[9]Actually the doubling property of the measure can also be weakened, see [21].

This notwithstanding, spaces supporting a Poincaré inequality already considerably generalize the class of admissible spaces for doing first order calculus. For further study of some weaker conditions, a discussion of some open questions, as well as a more extensive study of the different Sobolev type spaces, see [21] and [14] and the references therein.

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